PROBLEMS IN VIBRATION OF NONLINEAR THREADLINES

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "PROBLEMS IN VIBRATION OF NONLINEAR THREADLINES", submitted by LAWRENCE R. LAFRAMBOISE in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

The thesis deals with procedures used in solving two types of problems. The fixed end vibrating string is nonlinear in that the stress strain relation is perturbed from the usual elastic form to give a nonlinear equation of motion. The second problem deals with an elastic string which is moving through two fixed eyelets and the initial conditions are perturbed from a zero state. Expressions for the amplitude of transverse oscillations are derived and considerable attention is given to the period of time for which this solution exists and is valid.

The techniques used involve separating the problem into two separate problems both having characteristic parameters as independent variables. The solution in terms of characteristic coordinates is obtained by finding the Riemann invariants and expressing the derivatives of the amplitude of oscillation in terms of these. The mapping of the plane of the characteristic coordinates into the space-time plane is approximated by calculating first terms of power series expressions in the perturbation factor ε . That is, power series solutions are assumed, and the coefficients of the system are also expressed in power series, and then the equations are integrated.

A breakdown of the transformation from the characteristic to the physical planes is observed in that the Jacobian becomes zero. This indicates the boundary of the interval of existance of the solution, as well as the points where some of the derivatives of the amplitude become unbounded. The nature of the mappings at the point of breakdown are considered.

Numerical calculations are performed to determine the paths of the characteristic curves and thus determine the time and nature of breakdown in the case of the fixed string problem.

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CHAPTER I

INTRODUCTION

The purpose of this thesis is to invesitgate the solution of a number of almost linear hyperbolic partial differential equations in one dependent and two independent variables such as might describe the motion of vibrating strings. Many of the techniques employed, in particular, perturbation procedures, have been developed for problems in hydrodynamics. The basic idea is to approximate a given problem with a solvable linear problem and then to perturb this solution in terms of a small parameter. Solutions are expressed in terms of power series in this parameter. We shall attempt to correlate and explain various approaches to such problems in this thesis.

Breakdown of the method of solution, similar to that associated with the formation of shock waves in hydrodynamics, occurs in these problems. We study this both analytically and numerically.

The first problems to be considered deal with a vibrating string with fixed ends. The string is nonlinear in the sense that the linear stress-strain relation,

$$T = T_o(1 + Y_x) \quad , \tag{1.1}$$

does not hold, where as usual T denotes tension, T_o is equilibrium tension, Y is lateral displacement, x is the longitudinal coordinate, and subscripts denote partial differentiation.

Instead we replace the relation (1.1) by one of the relations

$$T = T_o(1 + Y_x + \frac{1}{2} \varepsilon Y_x^2) \quad , \tag{1.2}$$

or

$$T = T_o(1 + Y_x + \frac{1}{3} \varepsilon Y_x^3) \quad , \tag{1.3}$$

where ε is a small parameter. Then we obtain the equation of motion by substituting for T in

$$\rho Y_{tt} = T_x \quad , \tag{1.4}$$

where ρ is the density which we take as constant, and t is time.

Attention has recently been drawn to these equations as a result of numerical experiments carried out by E. Fermi, J. Pasta and S. Ulam [7] in the investigation of standing longitudinal oscillations of nonlinear beaded strings. We briefly describe the report [7].

Consider a string consisting of $L-1$ identical point masses, evenly distributed along the string from $x = 0$ to $x = \Pi$. We assume the masses are connected to each other by a weightless string and the i^{th} particle is situated at

$$x_i = \frac{i\Pi}{L} \quad , \qquad i = 1,2,\ldots,L-1 \quad .$$

Let y_i be the transverse displacement of the i^{th} particle. The equation of motion corresponding to (1.3) or (1.4) is

$$\frac{\partial^2 y_i}{\partial \tau^2} = (y_{i+1} + y_{i-1} - 2y_i) + \alpha[(y_{i+1} - y_i)^2 - (y_i - y_{i-1})^2]$$

or (1.5)

$$\frac{\partial^2 y_i}{\partial \tau^2} = (y_{i+1} + y_{i-1} - 2y_i) + \beta[(y_{i+1} - y_i)^3 - (y_i - y_{i-1})^3] \quad ,$$

respectively. Here α and β, the coefficients of the quadratic and cubic terms in the force between neighbouring particles, are assumed small, and $\tau = ct$ where c depends on the spacing between the particles. An analysis for various times t is attempted in terms of a finite Fourier series:

$$y_i = \sum_{k=1}^{L-1} A_k(t) \sin \frac{k\Pi i}{L} \quad . \tag{1.6}$$

To determine the coefficients $A_k(t)$ it is simpler to extend the string periodically to $[0,2\Pi]$. We can then use the relations

$$\sum_{i=1}^{2L-1} \sin \frac{k\Pi i}{L} \sin \frac{j\Pi i}{L} = \begin{cases} L & j = k \neq 0 \quad , \\ 0 & \begin{array}{l} j \neq 0 \; , \\ j = k \neq nL \qquad n = 0,1,\ldots \end{array} \end{cases} \tag{1.7}$$

so that

$$A_k(t) = \frac{1}{L} \sum_{i=1}^{2L-1} y_i \sin \frac{k\Pi i}{L} \quad . \tag{1.8}$$

In terms of the y_i variables, the total energy E_T is

$$
\begin{aligned}
E_T &= \frac{1}{2}(\dot{y}_1^2 + \ldots + \dot{y}_{2L-1}^2) \\
&\quad + \tfrac{1}{2}(2y_1^2 + \ldots + 2y_{2L-1}^2 - 2y_1y_2 - \ldots - 2y_{2L-1}y_{2L-2}) .
\end{aligned} \tag{1.9}
$$

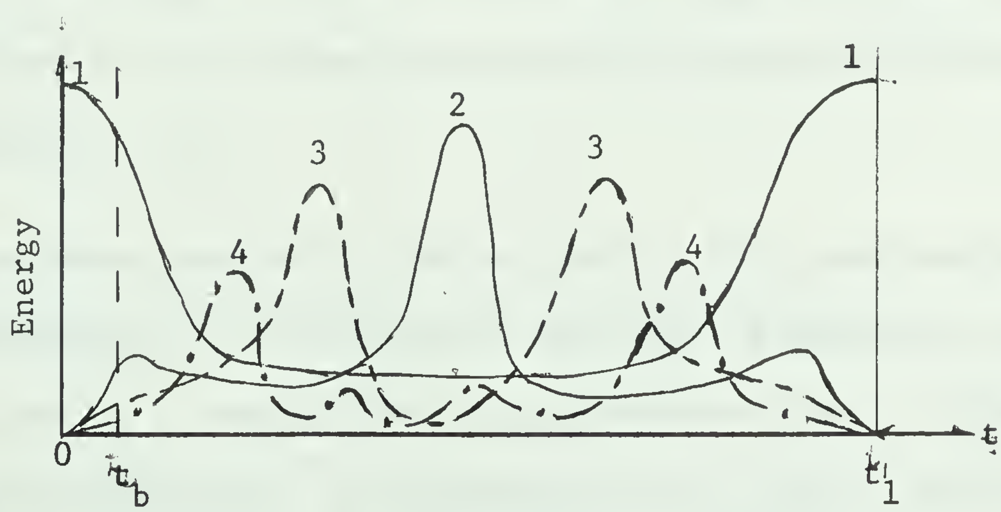
In terms of the A_k we have,

$$E_T = \frac{L}{2} \sum_{k=1}^{2L-1} \dot{A}_k^2 + 2L \sum_{k=1}^{2L-1} A_k^2 \sin^2 \frac{\Pi k}{2L} . \tag{1.10}$$

If we associate the term

$$\frac{1}{2} \dot{A}_k^2 + 2A_k^2 \sin^2 \frac{\Pi k}{2L} \tag{1.11}$$

with the k^{th} mode of vibration, this should indicate the extent to which the k^{th} mode is present. The conjecture of F.P.U.[1] was that the energy should be equipartitioned to the $2L-1$ modes. But an unusual recurrence pattern appeared in the total energy of various modes. After a certain length of time t_1, most of the energy appeared in the initial mode as depicted in Figure I.



ENERGY IN LOW MODES OF NONLINEAR STRING

Figure I

[1]F.P.U. refers to E. Fermi, J. Pasta, and S. Ulam with specific reference to their paper [7].

In an attempt to explain these phenomena, Zabusky and Kruskal [15], [17], investigated the continuous analogue of the first of the F.P.U. problems. The equations they considered are given by

$$\begin{aligned}
&Y_{tt} = (1 + \varepsilon Y_x) Y_{xx} , && \begin{cases} 0 < x < \ell \\ t > 0 \end{cases} \\
&\left.\begin{aligned} &Y(x,0) = a \sin \frac{\Pi x}{\ell} , \\ &Y_t(x,0) = 0 , \end{aligned}\right\} && \quad 0 \leq x \leq \ell \\
&Y(0,t) = Y(\ell,t) = 0 && \quad t \geq 0 \quad .
\end{aligned} \tag{1.12}$$

In their approach, to be considered in Chapter II, a breakdown of the solution was predicted at time $t_b = \frac{4}{\varepsilon a \Pi^2}$, which it appears is too small to permit an examination of the recurrence phenomenon. The solutions, which are in effect series expansions in ε , are for $t < t_b$ in agreement with the F.P.U. results.

A second estimate of the time of breakdown was arrived at by P. Lax [9], by deriving an ordinary differential equation involving the derivative of one of the Riemann invariants, and showing that this quantity becomes unbounded at t_b .

A procedure related to the one used in [17], and similar to methods introduced by C. Lin [13] and P. Fox [9], is employed in this thesis. It involves a mapping from an intermediate plane, a plane of "characteristic coordinates", to the physical x,t plane. First terms of a series representation of that transformation indicate the occurrence

of the breakdown. A solution to the second F.P.U. problem;

$$Y_{tt} = (1 + \varepsilon Y_x^2) Y_{xx} \qquad \begin{array}{l} 0 \le x \le \ell \\ t \ge 0 \end{array}$$

$$Y(0,t) = Y(\ell,t) = 0 \tag{1.13}$$

$$Y(x,0) = f(x) \ , \quad y_t(x,0) = 0$$

for ε sufficiently small, is found and the corresponding time of breakdown indicated[2].

The methods used in the fixed end vibrating string problem are then applied to another class of equations, those describing the "moving threadline" problem. Consider an elastic string or tape moving with constant velocity through fixed eyelets or rollers. If the line is displaced by a small amount it should begin to vibrate.

VIBRATING STRING PASSING THROUGH FIXED EYELETS

Figure II

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[2]Because of the breakdown of the transformation, the analysis of the above equations cannot yeild a result corresponding to the recurrence observed in the F.P.U. report. Kruskal and Zabuski have since published a paper dealing with a more complex equation whose solution does correspond to the results of that report. It appears in 'Interaction of Solutions in a Collisionless Plasma and the Recurrence of Initial States', Zabuski and Kruskal, Physical Review Letters, Vol. 15, 9 August, 1965.

It is of interest to determine both the motion of the line and the time when breakdown of the solution might be expected. Physically, breakdown might correspond to a snapping of the line. We consider two models describing this situation as developed by J. Zaiser [18] and presented by Ames [1]. These are given by

$$\frac{\alpha^2}{4} Y_{xx} + \alpha Y_{xt} + Y_{tt} = \frac{Y_{xx}}{1+Y_x^2} \quad , \qquad \text{I} \qquad (1.14)$$

$$\frac{\alpha^2}{4(1+Y_x^2)} Y_{xx} + \frac{\alpha}{(1+Y_x^2)^{\frac{1}{2}}} Y_{xt} + Y_{tt} = \frac{Y_{xx}}{1+Y_x^2} \quad , \qquad \text{II} \qquad (1.15)$$

$$Y(0,t) = Y(1,t) = 0 \quad , \qquad t \geq 0 \quad ,$$

where $\frac{\alpha}{2}$ denotes the ratio of the speed of the string to the speed of sound in the medium, t is time, Y is lateral displacement, and x is longitudinal coordinate. Equation (1.15) differs from (1.14) in that the longitudinal velocity of individual elements is varied, that is $\frac{\alpha}{2}$ is replaced by $\frac{\alpha}{2(1+Y_x^2)^{\frac{1}{2}}}$.

The perturbation parameter, in these problems, unlike the fixed end problems, does not occur in the equation of motion; we shall introduce it into the initial conditions as follows:

$$Y(x,0) = \varepsilon \sin \pi x \quad , \qquad 0 \leq x \leq 1 \quad ,$$
$$Y_t(x,0) = 0 \quad . \qquad (1.16)$$

Extensive numerical results are available for this problem in J. Zaiser's

thesis [18]. For values of t for which the calculations were carried out, no singularity in the solution or its derivatives was observed in [18].

In this thesis, expressions for the first terms of a solution in power series in ε are developed and it is shown that our solution does exhibit breakdown.

The breakdown is in fact predicted for a value of t_b covered by Zaiser's calculations.

We deal with two forms of pertubation: first, perturbation of the equations of the system and second, of the initial conditions. We show in Appendix I that these in general are equivalent.

We now include a brief discussion of the various approaches available to us. We are concerned with equations of the form:

$$aY_{xx} + bY_{xt} + cY_{tt} = 0 \tag{1.17}$$

in the half plane on $t > 0$, where Cauchy data is given on $t = 0$,

$$\begin{aligned} Y(x,0) &= \bar{f}(x) \\ & \qquad\qquad -\infty < x < +\infty \quad , \\ Y_t(x,0) &= g(x) \end{aligned} \tag{1.18}$$

with $\bar{f}(x)$ continuously differentiable. The coefficients a, b, c are functions of Y_x, Y_t only and perhaps of a parameter ε. Such a condition in the coefficients opens a number of approaches and is necessary to the ready solvability of our equations by the techniques we shall use.

Since the data for the actual problems we consider is given over a finite space interval $0 \le x \le 1$, it is necessary to extend our data by a continuation to the whole x axis.

We write the system as two first order equations by introducing two variables $u = Y_x$, $v = Y_t$, and writing the equivalent matrix form [4],

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + A(u,v) \begin{pmatrix} u \\ v \end{pmatrix}_x = 0$$

$$A = \begin{pmatrix} 0 & -1 \\ \frac{a}{c} & \frac{b}{c} \end{pmatrix} , \qquad c \neq 0 , \tag{1.19}$$

$$u(x,0) = f(x) ,$$

$$v(x,0) = g(x) , \qquad \text{where} \qquad f(x) = \frac{d}{dx} \bar{f}(x) .$$

The procedure used in this thesis consists in writing two pairs of differential equations for u,v and x,t with parameters ξ,η (characteristic variables) as independent variables. One might consider the use of a number of other techniques in the solution of the problems.

Because the coefficients are functions of u and v only, the hodograph transformation would seem useful in order to obtain a linear system. This involves writing the system with x,t as dependent variables of u,v. But the mapping of (u,v) into the (x,t) plane is many valued except in restricted ranges, which further complicates the assigning of boundary conditions. Although the coefficients are functions of the

independent variables, finding a solution is still not trivial.

Zabusky has solved the linear equation, obtained by a hodograph transformation, by the use of Riemann's method for the first fixed end problem [15]. He was able to write down the solution which is locally valid because the required Riemann function is known, and by an "unfolding process", the solution is extended to the half plane. However application of this procedure is limited by the scarcity of known Riemann functions [2].

An approach similar to the one we use consists in attempting a solution in the form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}^{0} + \varepsilon \begin{pmatrix} u \\ v \end{pmatrix}^{1} + \ldots \quad . \tag{1.20}$$

If we can write

$$A(u,v) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \ldots \quad , \tag{1.21}$$

with A_0 constant we obtain a sequence of problems

$$\begin{pmatrix} u \\ v \end{pmatrix}^{(o)}_t + A_o \begin{pmatrix} u \\ v \end{pmatrix}^{(o)}_x = 0 \quad ,$$

$$\begin{pmatrix} u \\ v \end{pmatrix}^{(n)}_t + A_o \begin{pmatrix} u \\ v \end{pmatrix}^{(n)}_x = - \sum_{j=1}^{n} A_j \begin{pmatrix} u \\ v \end{pmatrix}^{(n-j)}_x \quad , \qquad n \geq 1 \quad , \tag{1.22}$$

with initial conditions $u^{(n)}(x,o) = f^{(n)}(x)$, $v^{(n)}(x,o) = g^{(n)}(x)$, where $f(x) = \sum_{k=o}^{\infty} \varepsilon^k f^{(k)}(x)$ and $g(x) = \sum_{k=o}^{\infty} \varepsilon^k g^{(k)}(x)$. Since the left hand side of each equation is linear with constant coefficients, solutions of the sequence of problems may be carried out successively. The

difficulty with such an approach in problems where breakdown occurs is that it is difficult to obtain information in the neighbourhood of singular points because the solution series often do not converge there. The advantage of the method we use is that the series obtained converge up to and beyond the breakdown point [9].

CHAPTER II

PERTURBATION METHODS OF SOLUTION; CHARACTERISTIC VARIABLES

Consider the system

$$\underset{\sim}{u}_t + A\underset{\sim}{u}_x = 0 \quad , \tag{2.1}$$

where $A = A(\underset{\sim}{u})$ is a 2 by 2 matrix, and $\underset{\sim}{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ is a two-dimensional vector. The suffixes denote partial differentiation. We suppose that A has, in the region under consideration, two real and distinct eigenvalues P, Q, so that the system is hyperbolic. Let $\tilde{m}_i = (m_i, n_i)$, $i = 1,2$ be the real left eigenvectors corresponding to P, Q respectively, so that we have:

$$\begin{aligned} \tilde{m}_1 A &= P\tilde{m}_1 \\ \tilde{m}_2 A &= Q\tilde{m}_2 \quad . \end{aligned} \tag{2.2}$$

It immediately follows from equation (2.1) that

$$\begin{aligned} \tilde{m}_1 \underset{\sim}{u}_t + P\tilde{m}_1 \underset{\sim}{u}_x &= 0 \\ \tilde{m}_2 \underset{\sim}{u}_t + Q\tilde{m}_2 \underset{\sim}{u}_x &= 0 \quad , \end{aligned} \tag{2.3}$$

or equivalently that

$$\begin{aligned} m_1(u_t + Pu_x) + n_1(v_t + Pv_x) &= 0 \\ m_2(u_t + Qu_x) + n_2(v_t + Qv_x) &= 0 \quad . \end{aligned} \tag{2.4}$$

Because A is a function of u, v, ε only, so are P, Q, m_i, n_i, $i = 1,2$.

We now introduce the characteristic variables $\xi = \xi(x,y)$, $\eta = \eta(x,y)$. We assume a suitably smooth solution (u,v) and stipulate that along the curve ξ = constant that

$$x_\eta - Pt_\eta = 0 \quad , \tag{2.5a}$$

and along η = constant that

$$x_\xi - Qt_\xi = 0 \quad . \tag{2.5b}$$

We assume that we can do so in such a manner that the Jacobian is non-zero:

$$J = \frac{\partial(x,t)}{\partial(\xi,\eta)} \neq 0 \quad .$$

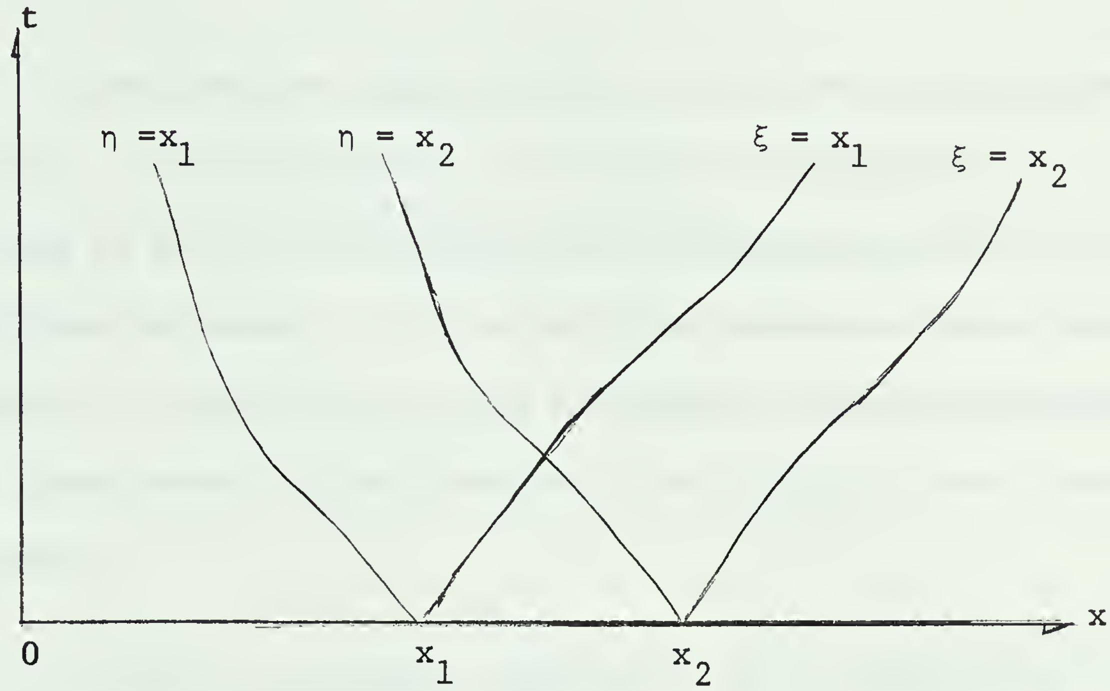
In this case, the equations (2.4) can be written in the form:

$$\begin{aligned} m_1 u_\eta + n_1 v_\eta &= 0 \quad \text{along } \xi = \text{constant} \\ m_2 u_\xi + n_2 v_\xi &= 0 \quad \text{along } \eta = \text{constant} \ . \end{aligned} \tag{2.6}$$

The equations (2.5), with the properly transformed initial conditions, describe the mapping from the (x,t) to the (ξ,η) plane while (2.6) describes that between the (u,v) and (ξ,η) planes.

We designate the characteristic variables by setting the parameters ξ,η equal to the x coordiante of the point at which the characteristic curves η = constant and ξ = constant interesect the

initial line, $t = 0$. In our work, the slopes P, Q of the characteristic lines in the (x,t) plane are non-zero. Thus the point of intersection of a particular line with the x-axis is unique.



CHARACTERISTIC LINES ξ = const., η = const.

IN THE (x,t) PLANE

Figure III

In this way, the initial conditions,

$$\begin{aligned} u(x,0) &= f(x) \qquad -\infty < x < \infty \quad , \\ v(x,0) &= g(x) \quad , \end{aligned} \tag{2.7}$$

may be rewritten in the form:

$$\begin{aligned} u(\xi,\xi) &= f(\xi) \quad , \\ v(\xi,\xi) &= g(\xi) \quad , \end{aligned} \tag{2.8}$$

$$x(\xi,\xi) = \xi$$
$$t(\xi,\xi) = 0 \quad . \tag{2.9}$$

In the problems dealt with here, as in the papers by Fox [9] and Lin [13], the coefficients of the system are functions of u, v and ε only so that (2.6) can be solved independently of (2.5). Methods which utilize the factor ε to perturb from a simpler state generally are powerful and useful, but we use the Riemann invariants associated with the characteristic lines because of the simplicity and directness of the method.

To obtain the Riemann invariants, let us suppose that we can find two non-zero functions

$$h_1 = h_1(u,v) \ , \qquad h_2 = h_2(u,v) \ , \tag{2.10}$$

and suppose further that there exist functions

$$r = r(u,v) \ , \qquad s = s(u,v) \ , \tag{2.11}$$

such that

$$r_u = h_1 m_1 \ , \qquad r_v = h_1 n_1 \ ,$$
$$s_u = h_2 m_2 \ , \qquad s_v = h_2 n_2 \quad . \tag{2.12}$$

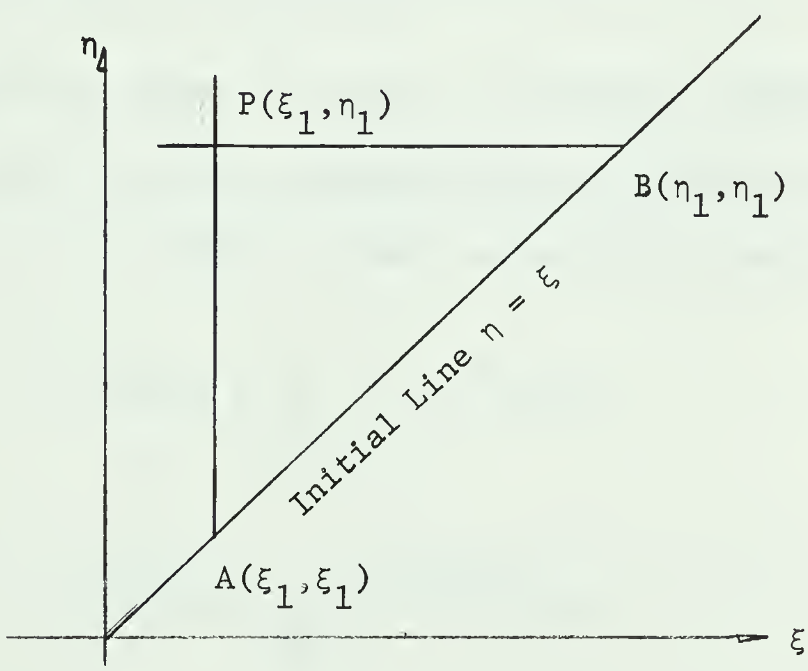
If this is possible, the system (2.6) becomes

$$
\begin{aligned}
r_\eta &= r_u u_\eta + r_v v_\eta \\
&= 0 \qquad \text{along} \quad \xi = \text{const.} \quad , \\
s_\xi &= s_u u_\xi + s_v v_\xi \\
&= 0 \qquad \text{along} \quad \eta = \text{const.} \quad ,
\end{aligned}
\tag{2.13}
$$

which shows r and s are constant along lines $\xi = \text{const.}$ and $\eta = \text{const.}$ respectively. Thus one concludes that

$$
\begin{aligned}
r &= r(u,v) = r(\xi) \quad , \\
s &= s(u,v) = s(\eta) \quad ,
\end{aligned}
\tag{2.14}
$$

and we can calculate r and s for points in the plane by doing so at the initial line.



CHARACTERISTIC LINES IN THE
(ξ,η) PLANE

Figure IV

Along A P ,

$$r = r(\xi_1) = r(u(\xi_1,\xi_1),\ v(\xi_1,\xi_1))$$
$$= r(f(\xi_1),\ g(\xi_1))$$
$$s = s(\eta_1) = s(f(\eta_1),\ g(\eta_1)) \quad . \tag{2.15}$$

Thus r and s are known for $P(\xi_1,\eta_1)$ and in general we have:

$$u(\xi,\eta) = u(r(\xi),\ s(\eta))$$
$$v(\xi,\eta) = v(r(\xi),\ s(\eta)) \quad . \tag{2.16}$$

The feasibility of this method, and its applicability to our problems depends on the fact that the coefficients are functions of u, v, ε only and that the required functions r, s, h_1 and h_2 exist.

When these cannot be found, an alternate approach is that of Fox [9] or Lin [13]. Again, assuming that coefficients of (2.6) are functions of u, v, ε only, one seeks solutions of the form

$$u(\xi,\eta) = \sum_{k=o}^{\infty} \varepsilon^k\, u^{(k)}(\xi,\eta)$$
$$v(\xi,\eta) = \sum_{k=o}^{\infty} \varepsilon^k\, v^{(k)}(\xi,\eta) \quad , \tag{2.17}$$

and express

$$m_i = \sum_{k=o}^{\infty} \varepsilon^k \, m_i^{(k)}(u,v) \quad ,$$

$$\tag{2.18}$$

$$n_i = \sum_{k=o}^{\infty} \varepsilon^k \, n_i^{(k)}(u,v) \quad ,$$

$$f(\xi) = \sum_{k=o}^{\infty} \varepsilon^k \, f^{(k)}(\xi) \quad ,$$

$$\tag{2.19}$$

$$g(\xi) = \sum_{k=o}^{\infty} \varepsilon^k \, g^{(k)}(\xi) \quad .$$

Then one substitute in (2.6) and attempts a solution to the successive problems,

$$\sum_{i+j=k} [m_1^{(i)} \, u_\eta^{(j)}(\xi,\eta) + n_1^{(i)} \, v_\eta^{(j)}(\xi,\eta)] = 0 \quad ,$$

$$\tag{2.20}$$

$$\sum_{i+j=k} [m_2^{(i)} \, u_\xi^{(j)}(\xi,\eta) + n_2^{(i)} \, v_\xi^{(j)}(\xi,\eta)] = 0 \quad ,$$

with initial conditions

$$u^{(k)}(\xi,\xi) = f^{(k)}(\xi) \quad ,$$
$$v^{(k)}(\xi,\xi) = g^{(k)}(\xi) \quad , \tag{2.21}$$

for $k = 0,1,2,\ldots$.

Once we have $u(\xi,\eta)$, $v(\xi,\eta)$, we can attempt a solution of (2.5). We substitute these expressions in P and Q to obtain $P(\xi,\eta,\varepsilon)$, $Q(\xi,\eta,\varepsilon)$ which may be in series form. We integrate the equations

$$\begin{aligned} x_\eta &= P(\xi,\eta,\varepsilon)t_\eta \quad , \\ x_\xi &= Q(\xi,\eta,\varepsilon)t_\xi \quad , \end{aligned} \tag{2.22}$$

with boundary conditions

$$\begin{aligned} x(\xi,\xi) &= \xi \quad , \\ t(\xi,\xi) &= 0 \quad , \end{aligned} \tag{2.23}$$

from the initial line $\eta = \xi$, along the curves $\xi = \text{const.}$ and $\eta = \text{const.}$ respectively, to the point (ξ,η) . Thus

$$\begin{aligned} x(\xi,\eta) - \xi &= \int_\xi^\eta P(\xi,\sigma,\varepsilon)\, t_\sigma(\xi,\sigma)\, d\sigma \\ x(\xi,\eta) - \eta &= \int_\eta^\xi Q(\sigma,\eta,\varepsilon)\, t_\sigma(\sigma,\eta)\, d\sigma \quad . \end{aligned} \tag{2.24}$$

If we assume that we let $\varepsilon \to 0$, then P and Q are constant; we have straight line characteristics and the solution is known exactly. This suggests that we attempt a power series in ε as did Fox and Lin. We express

$$\begin{aligned} P(\xi,\eta,\varepsilon) &= \sum_{k=o}^{\infty} \varepsilon^k P^{(k)}(\xi,\eta) \quad , \\ Q(\xi,\eta,\varepsilon) &= \sum_{k=o}^{\infty} \varepsilon^k Q^{(k)}(\xi,\eta) \quad , \end{aligned} \tag{2.25}$$

and seek solutions of the form:

$$x(\xi,\eta) = \sum_{k=o}^{\infty} \varepsilon^k x^{(k)}(\xi,\eta) \quad ,$$

$$t(\xi,\eta) = \sum_{k=o}^{\infty} \varepsilon^k t^{(k)}(\xi,\eta) \quad . \tag{2.26}$$

Initial conditions are expressed as

$$x^o(\xi,\xi) = \xi$$

$$x^{(k)}(\xi,\xi) = 0 \qquad k = 1,2,... \tag{2.27}$$

$$t^{(k)}(\xi,\xi) = 0 \qquad k = 0, 1,2,... \quad .$$

We substitute in (2.4) and assuming that the series converge uniformly in the region we consider, we find

$$\sum_{k=o}^{\infty} \varepsilon^k x^{(k)}(\xi,\eta) - \xi = \sum_{k=o}^{\infty} \sum_{j=o}^{\infty} \varepsilon^{k+j} \int_{\xi}^{\eta} P^{(j)}(\xi,\sigma) t_{\sigma}^{(k)}(\xi,\sigma) d\sigma \ ,$$

$$\sum_{k=o}^{\infty} \varepsilon^k x^{(k)}(\xi,\eta) - \eta = \sum_{k=o}^{\infty} \sum_{j=o}^{\infty} \varepsilon^{k+j} \int_{\eta}^{\xi} Q^{(j)}(\sigma,\eta) t_{\sigma}^{(k)}(\sigma,\eta) d\sigma \quad . \tag{2.28}$$

We equate coefficients of ε^k and assume that P^o and Q^o are constants. Thus,

$$x^o(\xi,\eta) - \xi = P^o t^o(\xi,\eta) \quad ,$$

$$x^o(\xi,\eta) - \eta = Q^o t^o(\xi,\eta) \quad , \tag{2.29}$$

$$x^{(k)}(\xi,\eta) = P^{o}t^{(k)}(\xi,\eta) + \int_{\xi}^{\eta} \sum_{n=1}^{k} P^{(n)}(\xi,\sigma)t_{\sigma}^{(k-n)}(\xi,\sigma)d\sigma \ ,$$

$$x^{(k)}(\xi,\eta) = Q^{o}t^{(k)}(\xi,\eta) - \int_{\xi}^{\eta} \sum_{n=1}^{k} Q^{(n)}(\sigma,\eta)t_{\sigma}^{(k-n)}(\sigma,\eta)d\sigma \ . \tag{2.30}$$

By addition and subtraction we obtain

$$x^{o}(\xi,\eta) = \frac{\eta+\xi}{2} + \frac{P^{o}+Q^{o}}{P^{o}-Q^{o}} \cdot \frac{\eta-\xi}{2}$$

$$t^{o}(\xi,\eta) = \frac{\eta-\xi}{P^{o}-Q^{o}} \tag{2.31}$$

$$t^{(k)}(\xi,\eta) = [Q^{o}-P^{o}]^{-1} \int_{\xi}^{\eta} \sum_{n=1}^{k} [P^{(n)}(\xi,\sigma)t_{\sigma}^{(k-n)}(\xi,\sigma)+Q^{(n)}(\sigma,\eta)t_{\sigma}^{(k-n)}(\sigma,\eta)]d\sigma \tag{2.32}$$

for $k = 1,2,\ldots,$ and $x^{(k)}$ can be calculated from (2.30).

In the problems we deal with, the assumptions made on P and Q are satisfied and we can choose an upper bound for $\eta-\xi$, so that, as is shown in Appendix III, these series for $x(\xi,\eta)$ and $t(\xi,\eta)$ converge for all required values of (ξ,η), with the condition that

$$|A + B + A' + B'| \leq |P^{o} - Q^{o}| \ , \tag{2.33}$$

where we assume that for the values of ε,ξ,η considered: $|P| \leq A$, $|Q| \leq B$, $|P_{\eta}| \leq A'$, $|Q_{\xi}| \leq B'$. Such conditions involve ε implicity. When more specific information is available, specific bounds on ε and on other parameters may be found; for example, see Fox [9].

We are particularly interested in solutions at values of t where

breakdown occurs. We see from (2.31) that t depends mainly on $\eta-\xi$, for which a large upper bound can be chosen.

Once we have the solutions

$$\begin{aligned} u &= u(\xi,\eta) \quad , \\ v &= v(\xi,\eta) \quad , \\ x &= x(\xi,\eta) \quad , \\ t &= t(\xi,\eta) \quad , \end{aligned} \tag{2.34}$$

then with an assumption that $\frac{\partial(x,t)}{\partial(\xi,\eta)} \neq 0$, it follows that we may solve for ξ,η in terms of x,t. Hence

$$\begin{aligned} u(x,t) &= u(\xi(x,t),\, \eta(x,t)) \\ v(x,t) &= v(\xi(x,t),\, \eta(x,t)) \end{aligned} \tag{2.35}$$

expresses solutions for Y_x, Y_t, from which Y can be calculated. We deal with this in the individual cases.

A slightly different approach is that used by Zabusky and Kruskal [17]. They assume that

$$\begin{aligned} P &= P^o + \varepsilon\overline{P} \quad , \\ Q &= Q^o + \varepsilon\overline{Q} \end{aligned} \tag{2.36}$$

where P^o and Q^o are constants. From (2.22) and (2.23) we have:

$$\begin{aligned} x(\xi,\eta) - \xi &= P^o\, t(\xi,\eta) + \varepsilon \int_\xi^\eta \overline{P}(\xi,\sigma,\varepsilon) t_\sigma(\xi,\sigma)d\sigma \\ x(\xi,\eta) - \eta &= Q^o\, t(\xi,\eta) - \varepsilon \int_\xi^\eta \overline{Q}(\sigma,\eta,\varepsilon) t_\sigma(\sigma,\eta)d\sigma \quad . \end{aligned} \tag{2.37}$$

Substracting we obtain:

$$t(\xi,\eta) = [P^o-Q^o]^{-1}\{(\eta-\varepsilon)-\varepsilon \int_{\xi}^{\eta} [\overline{P}(\xi,\sigma,\varepsilon)t_\sigma(\xi,\sigma)+\overline{Q}(\sigma,\eta,\varepsilon)t_\sigma(\sigma,\eta)]d\sigma\} \quad . \qquad (2.38)$$

An initial approximation

$$t^{(o)} = \frac{\eta - \xi}{P^o - Q^o} \qquad (2.39)$$

is assumed and substituting $t_\sigma^{(k-1)}$ in the right side we calculate t^k. Successive approximations for $k = 1,2,...$ are calculated. Convergence of this sequence is discussed in Appendix II.

If instead of choosing the value of ξ and η according to the characteristic lines x intercept, we let them equal their corresponding Riemann invariant, $\xi = r(\xi)$ and $\eta = s(\eta)$, then we can write (2.5) as

$$\begin{aligned} x_r - Pt_r &= 0 \quad , \\ x_s - Qt_s &= 0 \quad . \end{aligned} \qquad (2.40)$$

Differentiating with respect to s and r and subtracting we obtain

$$t_{rs} + \frac{P_s t_r}{P - Q} - \frac{Q_r t_s}{P - Q} = 0 \quad , \qquad (2.41)$$

an equation amenable to treatment by Riemann function methods (see Zabusky [15]).

CHAPTER III

BREAKDOWN

In our calculations up to this point, we have discussed solutions u,v on the understanding that u_x, v_x, u_t and v_t exist and are continuous in the required region. It was also assumed that the Jacobian $J = \frac{\partial(x,t)}{\partial(\xi,\eta)} \neq 0$ there. We have equations in the form:

$$\begin{aligned} x_\eta &= P\, t_\eta \qquad & x(\xi,\xi) &= \xi \\ x_\xi &= Q\, t_\xi \qquad & t(\xi,\xi) &= 0 \quad , \end{aligned} \tag{3.1}$$

where $P = P(\xi,\eta,\varepsilon)$ and $Q = Q(\xi,\eta,\varepsilon)$ are distinct and bounded for all values of ξ, η, and we assume we have found convergent series solutions for (3.1), in the form

$$\begin{aligned} x(\xi,\eta) &= \sum_{k=o}^{\infty} \varepsilon^k\, x^{(k)}(\xi,\eta) \quad , \\ t(\xi,\eta) &= \sum_{k=o}^{\infty} \varepsilon^k\, t^{(k)}(\xi,\eta) \quad . \end{aligned} \tag{3.2}$$

We can now express the Jacobian:

$$J = x_\xi\, t_\eta - x_\eta\, t_\xi \tag{3.3}$$

as

$$J = [P - Q]t_\xi\, t_\eta \quad .$$

If it should happen that $J = 0$ at some point (ξ,η) then clearly our transformation from the (x,t) to the (ξ,η) plane is singular and either

$$t_\xi = x_\xi = 0 \ , \qquad \text{or} \qquad t_\eta = x_\eta = 0 \ ,$$

in which case ξ_t, ξ_x or η_t, η_x are unbounded. Thus $u_x = u_\xi \xi_x + u_\eta \eta_x$, u_t, v_x and v_t become unbounded and we cannot admit u,v as solution to our problem at such points. Points or curves along which such a break-down occurs define a boundary of the domain in which we have found a solution to our initial value problem. In this Chapter we examine the occurrence of such curves and the resultant "folding" of the characteristic plane.

From (3.3), we see that a locus of points in the (ξ,η) plane where $J = 0$ is obtained from the curves

$$\begin{aligned} & t_\xi(\xi,\eta) = 0 \ , \\ \text{or} \quad & \\ & t_\eta(\xi,\eta) = 0 \ . \end{aligned} \tag{3.4}$$

If we approximate

$$t_\xi = \sum_{k=o}^{\infty} \varepsilon^k t_\xi^{(k)}(\xi,\eta) = 0 \ , \tag{3.5}$$

by taking

$$t_\xi^0(\xi,\eta) + \varepsilon\, t_\xi^{(1)}(\xi,\eta) = 0 \ , \tag{3.6}$$

then we will approximate points at which (3.4) is satisfied. The assumption here is that we have

$$\left| \sum_{k=2}^{\infty} \varepsilon^k t_\xi^k(\xi,\eta) \right| < K\,\varepsilon \tag{3.7}$$

for some constant K, in the neighbourhood of points which satisfy (3.6). Thus, if ε is small, the solution to (3.6) cannot differ significantly from that of $t_\xi(\xi,\eta) = 0$. The analysis is similar for $t_\eta(\xi,\eta) = 0$.

We have, as yet, been unable to form a general theorem under which our assumption (3.7) can be shown to hold. It would appear that we must consider each problem individually. If one refers to Zabusky, where a specific Riemann function is known, and hence a specific representation of the solution $t(\xi,\eta)$ it is possible to verify these assumptions. One such verification is considered in Appendix 4.

As will be shown, in the problems we consider, it is because we can write

$$\begin{aligned} P^{(1)}(\xi,\eta) &= h_1(\xi) + g_1(\xi,\eta) \quad , \\ Q^{(1)}(\xi,\eta) &= h_2(\eta) + g_2(\xi,\eta) \quad , \end{aligned} \tag{3.8}$$

where h_i, g_i are bounded, and h_i, $i = 1,2$, are non-constant, that singularities occur. To show this we assume that P^o and Q^o are distinct constants and that

$$P = \sum_{k=o}^{\infty} \varepsilon^k P^{(k)}(\xi,\eta) \quad \text{and} \quad Q = \sum_{k=o}^{\infty} \varepsilon^k Q^{(k)}(\xi,\eta) \ . \tag{3.9}$$

From (2.32) we see

$$t^1_\xi = [Q^o-P^o]^{-1} \frac{\partial}{\partial\xi} \int_\xi^\eta [P^{(1)}(\xi,\sigma)t^o(\xi,\sigma)+Q^{(1)}(\sigma,\eta)t^o(\sigma,\eta)]d\sigma$$

$$= [Q^o-P^o]^{-2} \{-\int_\xi^\eta P_\xi^{(1)}(\xi,\sigma)d\sigma + Q^{(1)}(\xi,\eta)-P^{(1)}(\xi,\eta)\}$$

$$= [P^o-Q^o]^{-2} \{-\int_\xi^\eta \frac{\partial}{\partial\xi} h_1(\xi)d\sigma - \int_\xi^\eta \frac{\partial}{\partial\xi} g_1(\xi,\sigma)d\sigma + P^{(1)}(\xi,\eta)-Q^{(1)}(\xi,\eta)\}$$

$$= [P^o-Q^o]^{-2} \{-(\eta,\xi)h_1'(\xi) + B(\xi,\eta)\} \qquad (3.10)$$

where $B(\xi,\eta)$ represents terms uniformly bounded in ξ,η, $h_1'(\xi)$ represents $\frac{d}{d\xi} h_1(\xi)$ and we have assumed that $\int_\xi^\eta \frac{\partial}{\partial\xi} g_1(\xi,\sigma)d\sigma$ is uniformly bounded in ξ,η.

Thus, when we approximate breakdown by solving the equation

$$0 = t^o_\xi + \varepsilon\, t^{(1)}_\xi \quad , \qquad (3.11)$$

we have

$$0 = [P^o-Q^o]^{-2} \{Q^o-P^o-\varepsilon(\eta-\xi)h'(\xi)\} + \frac{\varepsilon B(\xi,\eta)}{[P^o-Q^o]} \quad .$$

or

$$0 = P^o-Q^o + \varepsilon(\eta-\xi)\, h_1'(\xi) + O(\varepsilon) \quad . \qquad (3.12)$$

In view of (3.7) we are in fact setting $t_\xi = 0 + O(\varepsilon)$ so that, in setting

$$\eta = \eta(\xi) = \xi - \frac{P^o-Q^o}{\varepsilon h_1'(\xi)} \quad , \quad \xi \in \{\sigma | (P^o-Q^o)h_1'(\sigma) < 0\} \quad , \qquad (3.13)$$

we are approximating the curve at which $t_\xi = 0$. A different approach to the study of the singularities is given by Zabusky and Kruskal [17]. Since

$$J = \frac{\partial(x,t)}{\partial(\xi,\eta)} = \frac{\partial(x-P^o t,\ x-Q^o t)}{\partial(\xi,\eta)} \cdot \frac{1}{[P^o-Q^o]} \quad , \tag{3.14}$$

where P^o and Q^o are distinct constants and

$$P = P^o + \varepsilon\overline{P} \ , \qquad Q = Q^o + \varepsilon\overline{Q} \quad ,$$

we can consider a different Jacobian. The derivatives involved here express the rate of deviation from the linear characteristics on which we are perturbing. If we write (2.36) in the form

$$\begin{aligned} x - P^o t &= \xi + \text{Ⓟ} \quad , \\ x - Q^o t &= \eta + \text{Ⓠ} \quad , \end{aligned} \tag{3.15}$$

where

$$\text{Ⓟ} = \int_\xi^\eta \overline{P}(\xi,\sigma,\varepsilon)\ t_\sigma(\xi,\sigma)\ d\sigma \quad ,$$

and (3.16)

$$\text{Ⓠ} = \int_\xi^\eta \overline{Q}(\xi,\eta,\varepsilon)\ t_\sigma(\sigma,\eta)\ d\sigma \quad ,$$

then the new Jacobian has the form

$$(1 + \text{Ⓟ}_\xi)(1 + \text{Ⓠ}_\eta) - \text{Ⓟ}_\eta \text{Ⓠ}_\xi \quad . \tag{3.17}$$

Since $\text{Ⓟ}_\eta\ \text{Ⓠ}_\xi$ is of order ε^2 , an approximation to the

location of zeros of J should be provided by setting:

$$\begin{aligned} 1 + Ⓟ_{\xi} &= 0 + O(\varepsilon^2) \\ 1 + Ⓠ_{\eta} &= 0 + O(\varepsilon^2) \quad . \end{aligned} \tag{3.18}$$

The procedure is now essentially the same as above.

Finally, a prediction of the time when singularities occur can be arrived at without actually solving the problems. The method, which involves examining the derivatives r_x, s_x of the Riemann invariants, was applied by P. Lax [12] to the problem

$$\begin{aligned} Y_{tt} &= \frac{1}{c^2}(1 + Y_x)Y_{xx} \qquad t > 0 \\ Y(x,0) &= b \sin \frac{\Pi x}{\ell} \quad , \end{aligned} \tag{3.19}$$

to predict a breakdown time, t_b , corresponding to those obtained by Zabusky and Kruskal.

Consider a set of Riemann invariants r and s of

$$Y_{tt} = K^2 Y_{xx} \tag{3.20}$$

and let f' and $f`$ denote differentiation of a function f along the characteristic directions. Let P,Q be distinct characteristic values of the matrix A in (2.1). Choose h such that:

$$h_s = P_s/P-Q \ , \qquad P_s = \frac{\partial P}{\partial s} \quad . \tag{3.21}$$

If we set

$$z = r_x \, e^h \quad , \tag{3.22}$$

$$a = -e^{-h} P_r \quad , \tag{3.23}$$

where $P_r = \frac{\partial P}{\partial r}$ is assumed negative, we obtain

$$z' + a z^2 = 0 \quad .$$

Two theorems are given in Lax's paper [12]:

"Theorem 1. Let $z(t)$ be the solution of the initial value problem

$$\frac{dz}{dt} = a(t)z^2 \qquad z(0) = m \tag{3.24}$$

in the interval $[0,T]$. Suppose that the function $a(t)$ satisfies the inequality

$$0 < B < a(t) \qquad 0 \leq t \leq T \; ,$$

and suppose that m is positive, then $T < (mB)^{-1}$.

Theorem 2. Suppose that $a(t)$ satisfies the inequality $|a(t)| < C$; then the initial value problem (3.24) has a solution for $|t| < |mC|^{-1}$."

If we assume that there is a positive maximum value , m_1 , of z on the initial line, and provided the bounds B and C exist, then the theorems to determine bounds for the time t_b , namely,

$$(Cm_1)^{-1} < t_b < (Bm_1)^{-1} \quad , \tag{3.25}$$

at which z is unbounded. Because $z = e^h r_x$, and h is bounded, r_x is also unbounded at that time.

To see how this applies to the present problem we let $K^2 = \frac{1}{c^2}(1+\varepsilon Y_x)$ in equation (3.20) and as in Chapter II we write the system in terms of $u = Y_x$, $v = Y_t$. The characteristic values P, Q are $-K$, K and we find Riemann invariants r, s such that $r_u = K$, $r_v = 1$, $s_u = -K$, $s_r = 1$; so that

$$r = v + L(u) \;, \quad s = v - L(u) \quad , \tag{3.26}$$

where $\frac{d}{du} L(u) = K$. In order to compute P_r , we differentiate

$$r - s = 2\,L(v) \;, \tag{3.27}$$

to obtain

$$1 = 2\,L(u)_u\, v_r = 2\,K\, u_r \quad . \tag{3.28}$$

Thus

$$\begin{aligned} P_r &= P_u\, u_r \\ &= \frac{-K_u}{2K} \\ &= \frac{-\varepsilon}{4}(1+ Y_x)^{-\frac{1}{2}} \;, \qquad |Y_x| < \frac{1}{e} \;, \end{aligned} \tag{3.29}$$

Further,

$$r_x = v_x + L_u\, u_x = v_x + K\, u_x$$

so that the initial values are

$$r_x(0) = \frac{1}{c}(1+\varepsilon Y_x(0))^{\frac{1}{2}}\, Y_{xx}(0) \quad . \tag{3.30}$$

Here (0) denotes the initial point of a given characteristic line.

Under the assumption that initial conditions have small variation, we can assume that the quantities P_r and h vary little from their initial values $P_r(0)$ and $h(0)$, so that one can estimate:

$$(Cm_1)^{-1} \doteq [\max (-e^{-h(0)} P_r(0)) \cdot \max (r_x(0)\ e^{h(0)})]^{-1}$$

$$= [\max (-P_r(0)) \cdot \max (r_x(0))]^{-1}$$

$$= \left[\frac{\varepsilon}{4} (1-\varepsilon\frac{b\Pi}{\ell})^{\frac{1}{2}} \cdot \frac{(1+\varepsilon\frac{b\Pi}{\ell})^{-\frac{1}{2}}}{c} \cdot \frac{b\Pi^2}{\ell^2} \right]^{-1} \quad . \qquad (3.31)$$

If we assume that ε is small, we have

$$(Cm_1)^{-1} = \frac{4\ \ell^2}{\varepsilon\Pi^2 bc} \quad . \qquad (3.32)$$

Similarly $(Bm_1)^{-1}$ has the same value, so that we have a prediction of breakdown time $t_b = \dfrac{4\ell^2}{cb\varepsilon\Pi^2}$.

But for $K = \frac{1}{c}(1+\varepsilon Y_x^p)^{\frac{1}{2}}$, $p = 2,3,\ldots$, we have

$$P_r = \frac{\varepsilon^2 p\ Y_x^{p-1}}{4(1+\varepsilon Y_x^p)^{\frac{1}{2}}}$$

$$= 0 \quad \text{for} \quad Y_x = 0 \qquad (3.33)$$

so that there exists no lower bound B for $|e^h P_r|$ and the theory does not provide an upper bound for the interval of existence. Thus breakdown is not predicted. For a more detailed investigation of the correspondence between the (ξ,η) and the (x,t) planes in the neighbourhood of the singular

lines we can refer to theorems due to Cragg [5], formulated for the study of the breakdown of the hodograph transformation.

From [5] we shall note two theorems. Let us first of all define a singular line as a line along which $J = 0$. Denote it by S in the (ξ,η) plane and by L in the (x,t) plane. We define a singular point of order n to be a point at which all derivatives of J of order less than n are zero as well as J . We discuss the singular line S , corresponding to $t_\xi = 0$.

We quote from [5], where we have replaced the symbols used there by the corresponding ones as used in this thesis.

"Theorem 1. Let a curve S in the (ξ,η) plane, each point of which is singular of order one, correspond to a curve L in the (x,t) plane. Then, save in exceptional cases, with each point of S is associated a privileged direction, such that to any curve crossing S in that direction there corresponds a curve having a singular point (in the usual geometrical sense) on L , while to any curve crossing S in a direction other than the privileged direction there corresponds a curve touching L .

Thoerem 2. If a cusp of a curve is defined as a singularity such that the adjacent parts of the curve on either side of it have the same tangent but in opposite senses, then (i) except where the singular line S is itself in the privileged direction, all curves crossing S in the privileged direction correspond to the curves cusped on L , and (ii) when the singular line S is itself in the privileged direction at a point and,

in the notation of Theorem 1, $t_{\xi\xi} + f''(\xi)t_{\alpha} \neq 0$, the limit line L is cusped ... .

Theorem 3. All points in the vicinity of a singular line S of order one correspond to points on one side of the limit line L ."

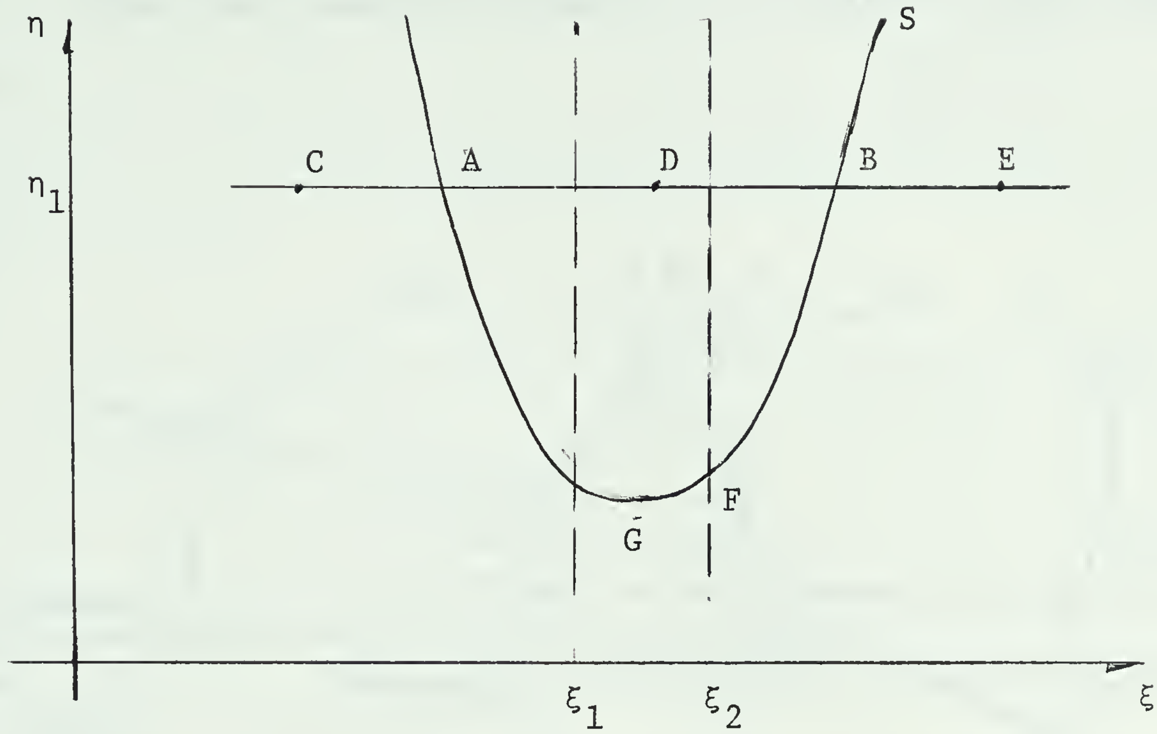
In the development of these theorems, it is shown that, corresponding to curves where $t_{\xi} = x_{\xi} = 0$, the curves η = const. are not in general, in the "privileged direction" and thus tangent to L in the (x,t) plane, also that L is cusped in the (x,t) plane at points where $t_{\xi\xi} = 0$. The parameter α, generally different from η or ξ, is that for which $\frac{\partial x}{\partial \alpha} = 0$ at a given singular point under consideration, and f is defined such that $\alpha = f(\xi)$ is the singular curve S .

From (3.12) we see that $t_{\xi\xi} = 0 + 0(\varepsilon)$ when

$$\varepsilon(\eta-\xi) \frac{d^2}{d\xi^2} h_1(\xi) = 0 \quad . \tag{3.34}$$

which corresponds to the minimum of $\eta(\xi)$, (3.13), where $\frac{d}{d\xi}\eta(\xi) = 0$. We do not consider points where singular lines interesect in the (ξ,η) plane, that is, where $t_{\xi} = t_{\eta} = x_{\xi} = x_{\eta} = 0$.

Consider Figures V and VI. Theorems 1 and 2 imply then that the initial breakdown point G , the extremum of the $t_{\xi} = 0$ line, where $t_{\xi\xi} = 0$, maps into a cusp in the (x,t) plane. The lines, η = const., and ξ = const. are tangent to L at points such as A, B and F . Theorem 3 implies that points C and D should be on the same side of L , and similarly for D and E .



LINES CROSSING S IN THE (ξ,η) PLANE

Figure V

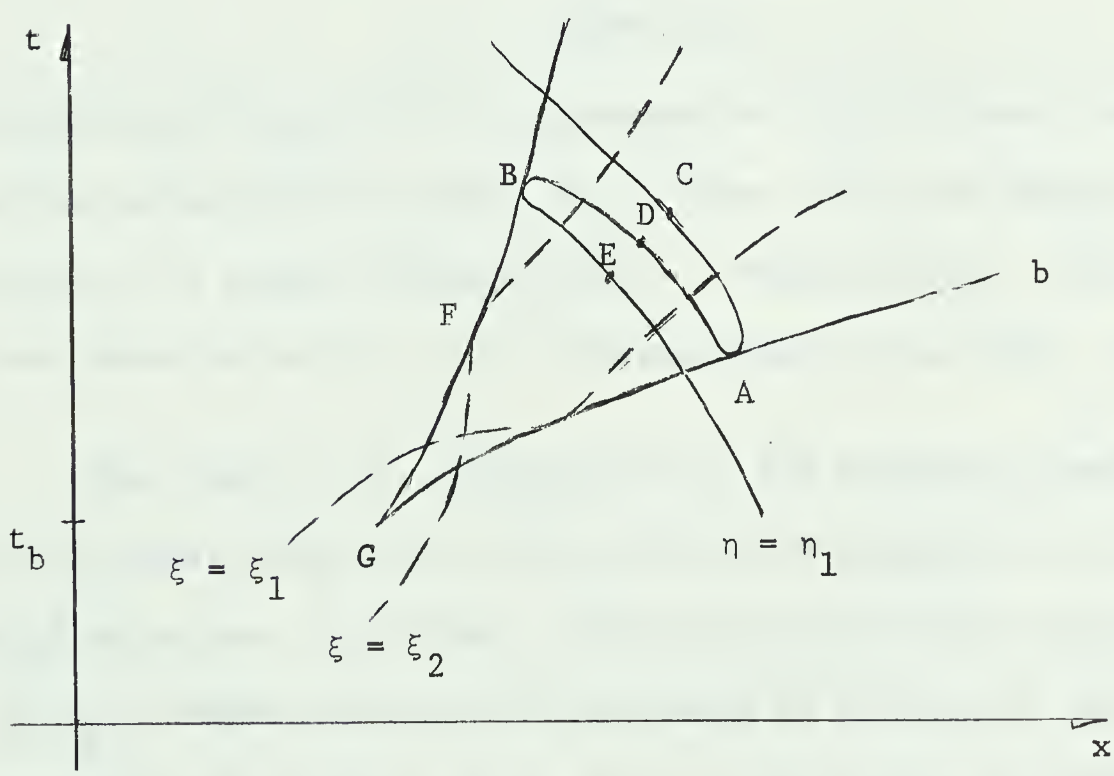
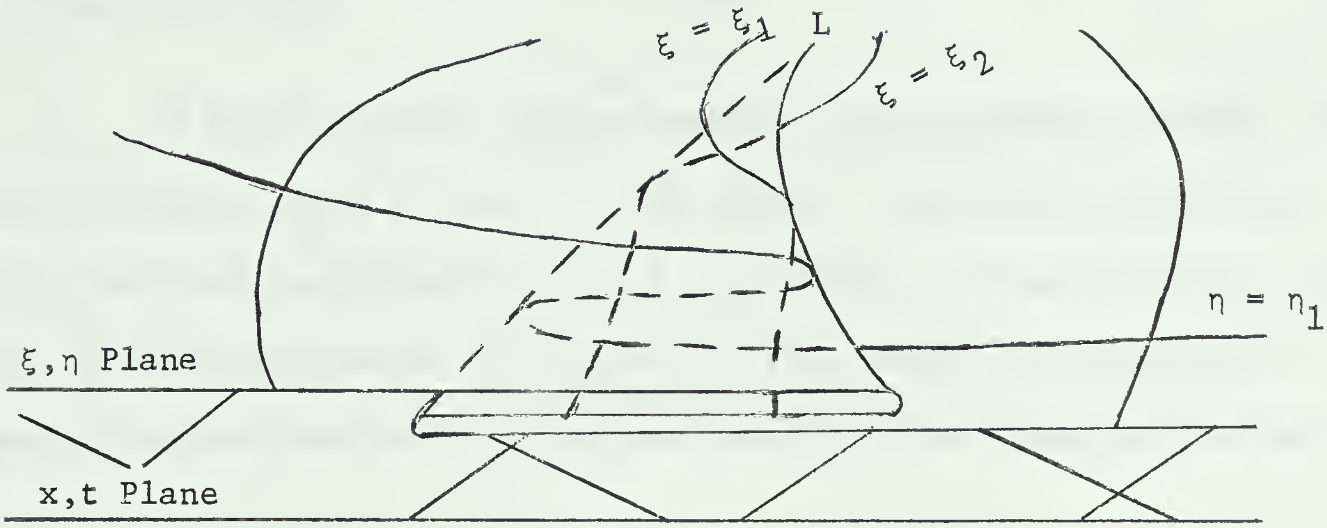


IMAGE LINES CROSSING L IN THE (x,t) PLANE

Figure VI

From this comes the notion of the "folding" of the characteristic plane.



FOLDING OF (ξ,η) PLANE ABOVE CORRESPONDING POINTS OF (x,t) PLANE

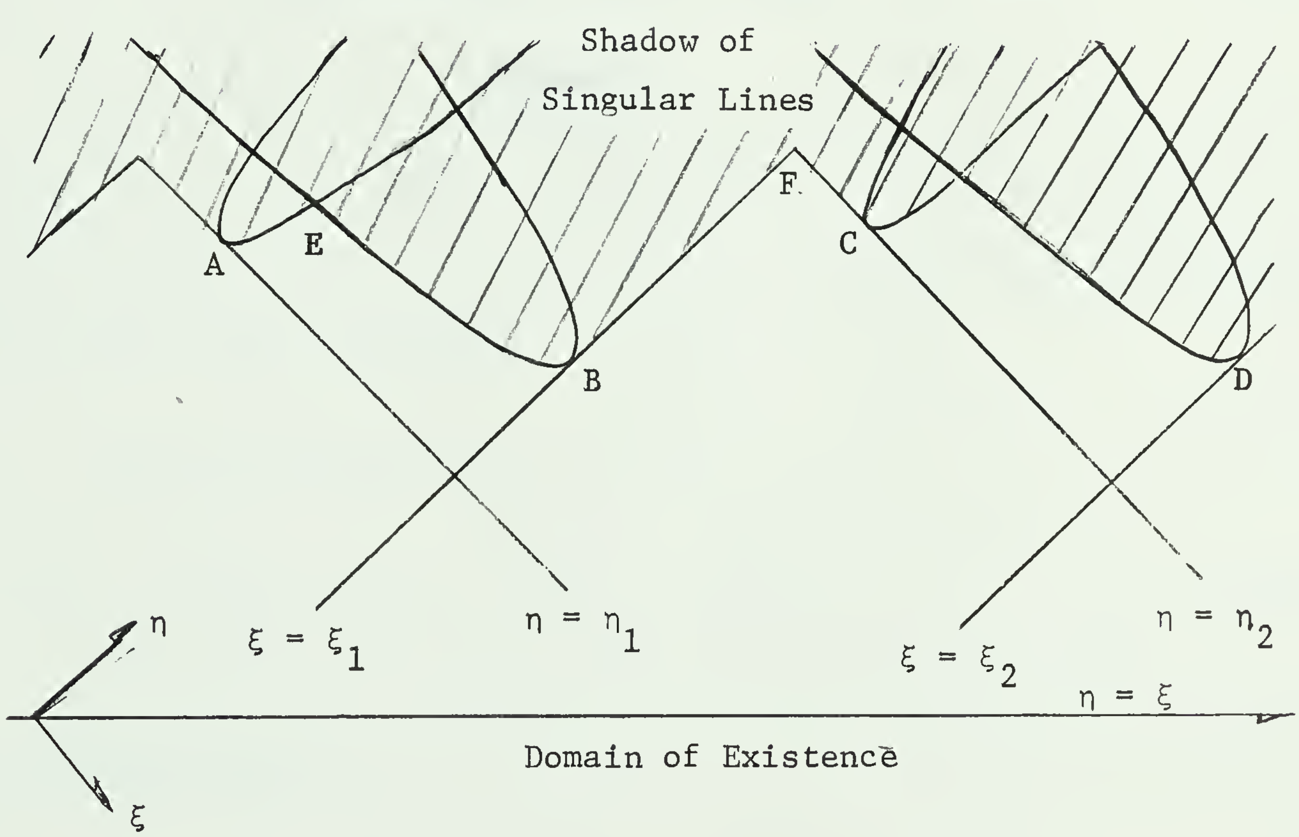
Figure VII

The folded sheet (Figure VII), represents the (ξ,η) plane placed above the corresponding points in the (x,t) plane. For each point of the (ξ,η) plane there is a unique solution (u,v) . Thus where the "fold" occurs there are three values of (ξ,η) corresponding to one point (x_1,t_1) .

This leads to no contradiction of the existence theorem [4], as only one of those three points lies between the singular line and the initial line in the (ξ,η) plane. While there are three values of (u,v) for (x_1,t_1) inside the region of existence of a solution, only the value corresponding to the point in the (ξ,η) plane between the initial line and the singular line can be considered a solution. Those points (ξ,η) in the "shadow" of the singular line are not in the domain of existence of the solution because, contrary to the existence theorem, u_x,v_x,u_t,v_t

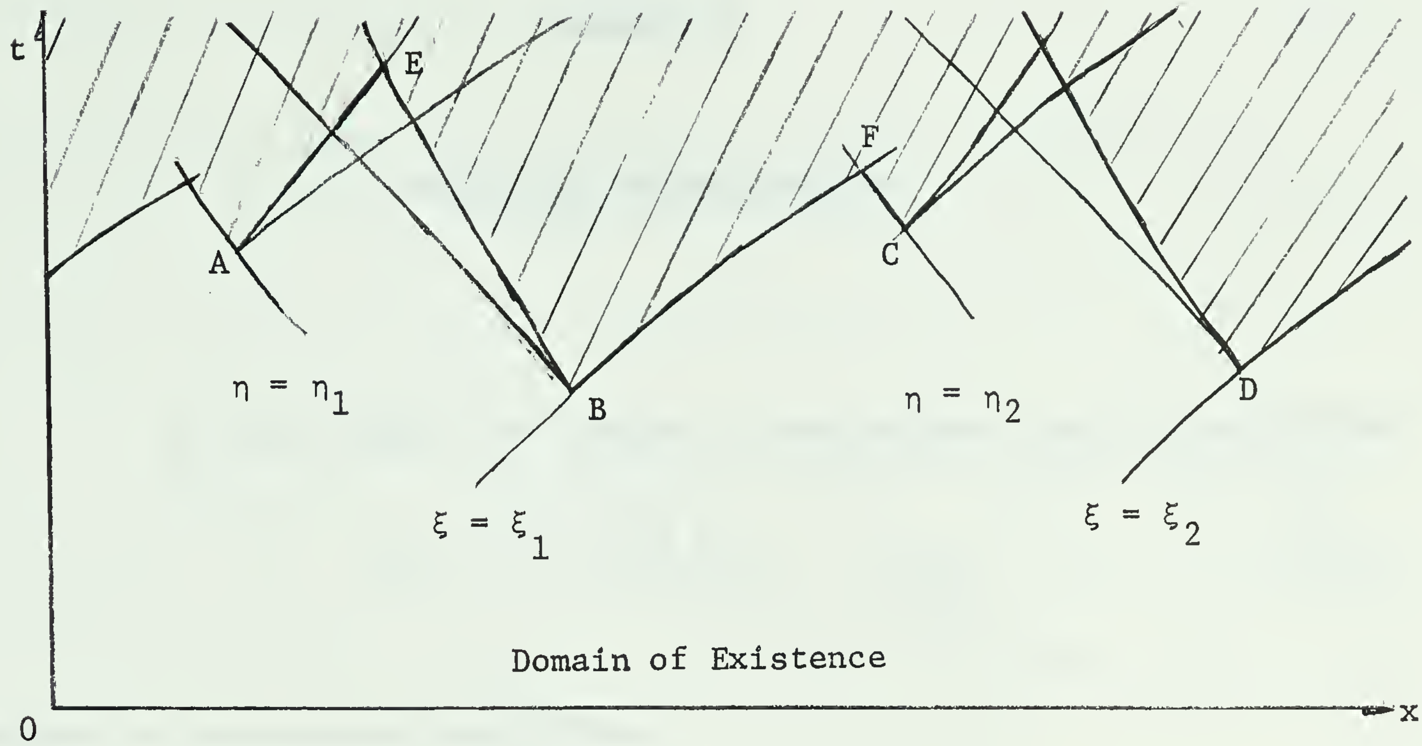
are unbounded in the domain of dependence of the point. Thus the upper boundary of the domain of existence consists of singular lines, (e.g. EB in Figures VIII and IX) and characteristic lines running up from the vertex of a cusp (e.g. BF) .

It will be seen, when we consider the particular problems, that the conditions $t_{\xi\xi} = 0$ and $t_{\eta\eta} = 0$ hold at the initial breakdown points and that the shape of $t_{\eta} = 0$ is similar to that in Figures V and VII. The characteristics of Figures VI, VII, and IX are exagerated, compared to the numerical results performed for the fixed end problem.



EXAMPLE OF DOMAIN OF EXISTENCE IN (ξ,η) PLANE

Figure VIII



EXAMPLE OF CORRESPONDING DOMAIN OF

EXISTENCE IN (x,t) PLANE

Figure IX

CHAPTER IV

THE FIXED STRING PROBLEM

In this chapter, we consider a generalized fixed string problem

$$Y_{tt} = (1 + \varepsilon Y_x^p)^q \, Y_{xx} \qquad \begin{array}{l} 0 < x < 1 \\ t > 0 \quad , \end{array} \tag{4.1}$$

subject to the boundary conditions,

$$\left.\begin{array}{l} Y(x,0) = a \sin n\Pi x \; , \\ Y_t(x,0) = 0 \quad , \end{array}\right\} \quad 0 \le x \le 1 \quad , \tag{4.2}$$

$$Y(0,t) = Y(1,t) = 0 \quad , \qquad t > 0 \quad ,$$

where n and p are positive integers and q is a non-zero integer. We will of course require that $1 + \varepsilon Y_x^p > 0$ in order to ensure boundedness in the various cases we consider.

We write the system as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ -\Phi^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; , \qquad \begin{array}{l} t > 0 \\ -\infty < x < \infty \end{array} \tag{4.3}$$

with initial conditions

$$\begin{array}{ll} u = U(x,0) = an\Pi \cos n\Pi x \; , & -\infty < x < \infty \\ v = V(x,0) = \;\; 0 \quad , & \end{array} \tag{4.4}$$

where

$$\Phi(u) = (1 + \varepsilon u^p)^{q/2} \quad . \tag{4.5}$$

It will be shown later that this periodic extension of initial data satisfies end conditions. We bound a and ε so that $0 < \varepsilon(na\Pi)^p < 1$; this will mean that $(1 + \varepsilon u^p) > 0$, which provides for the boundedness of $\Phi(u)$ and its derivatives.

Eigenvalues for the matrix are $P = -\Phi$, $Q = \Phi$, with left eigenvectors $(m_1,n_1) = (\Phi,1)$ and $(m_2,n_2) = (-\Phi,1)$. As in Chapter II, we express the system in terms of characteristic variables. Thus we have

$$\begin{aligned} -\Phi u_\xi + v_\xi &= 0 \quad , \\ \Phi u_\eta + v_\eta &= 0 \quad , \end{aligned} \tag{4.6}$$

$$\begin{aligned} x_\xi + \Phi t_\xi &= 0 \quad , \\ x_\eta - \Phi t_\eta &= 0 \quad , \end{aligned} \tag{4.7}$$

with boundary conditions,

$$\begin{aligned} u(\xi,\xi) &= f(\xi) \ , & x(\xi,\xi) &= \xi \quad , \\ v(\xi,\xi) &= 0 \quad , & t(\xi,\xi) &= 0 \quad , \end{aligned} \tag{4.8}$$

where $f(\xi) = an\Pi \cos n\Pi\xi$.

We employ the Riemann invariants to solve for $u(\xi,\eta)$, $v(\xi,\eta)$. We select $h_1 = -\frac{1}{2}$, $h_2 = \frac{1}{2}$ so that

$$\begin{aligned} r_u &= \frac{1}{2}\Phi(u) & r_v &= \frac{1}{2} \quad , \\ s_u &= \frac{1}{2}\Phi(u) & s_v &= -\frac{1}{2} \quad . \end{aligned} \tag{4.9}$$

The Riemann invariants are:

$$r = r(\xi) = \frac{1}{2} \int_o^u \Phi(\sigma)d\sigma + \frac{v}{2} ,$$
$$s = s(\eta) = \frac{1}{2} \int_o^u \Phi(\sigma)d\sigma - \frac{v}{2} . \qquad (4.10)$$

Let us consider first the simpler case where $p = 1$, $q \neq -2$, then,

$$\begin{aligned} r + s &= \int_o^u \Phi(\sigma)d\sigma \\ &= [\varepsilon(\frac{q}{2} + 1)]^{-1} \cdot [(1 + \varepsilon u)^{\frac{q}{2}} - 1] , \qquad (4.11) \\ r - s &= v . \end{aligned}$$

In general if we assume that $|u| < \frac{1}{\varepsilon^p}$ then we have $\frac{\partial(r,s)}{\partial(u,v)} = \frac{1}{2}\Phi \neq 0$ and we can invert the expressions so that, in this case with $p = 1$,

$$u = u(r+s) = \frac{1}{\varepsilon} [\varepsilon(\frac{q}{2} + 1)(r+s) + 1]^{1/(1 + \frac{q}{2})} - \frac{1}{\varepsilon}$$
$$v = s - r . \qquad (4.12)$$

Under the assumption that we are dealing with a region where u,v exist and are properly bounded, we have $r = r(\xi)$ and $s = s(\eta)$, constant along characteristic lines, and whose values we obtain from the initial line:

$$s(\sigma) = r(\sigma) = [2\varepsilon (\frac{q}{2} + 1)]^{-1} \cdot [(1 + \varepsilon f(\sigma))^{1 + \frac{q}{2}} - 1] . \qquad (4.13)$$

Combining (4.12) and (4.13) we obtain:

$$\begin{aligned} u(\xi,\eta) &= u(r(\eta),\ s(\xi)) \quad , \\ v(\xi,\eta) &= r(\xi) - s(\eta) \quad . \end{aligned} \tag{4.14}$$

Because, as we see in (4.9), r_u and s_u are positive, we have

$$\begin{aligned} \max r &= r(u_1) \\ \max s &= s(u_1) \end{aligned} \tag{4.15}$$

where $u_1 = \max u$, thus

$$\max\,(r+s) = r(u_1) + s(u_1) \quad . \tag{4.16}$$

If we set $r+s = \mu$ and consider (4.11), we see that $\frac{du}{d\mu} = \Phi(u) > 0$ and thus $\frac{du}{d\mu} = \Phi(u)^{-1} > 0$, for properly bounded values of u . Because u is monotonically increasing in $r + s$, $\max u = u(\max\,(r+s)) = u(r(u_1) + s(u_1))$. Thus the maximum value of u corresponds to the maximum value u_1 along the initial line and thus $u < \frac{1}{\varepsilon}$. By similar analysis we have a lower bound. Our solution $u(\xi,\eta)$ in (4.14) satisfies the condition $|u| < \frac{1}{\varepsilon}$ for all values of ξ and η .

We proceed with the general case where we will, in addition, have to resort to series expansions. Consider

$$r+s = \int_o^u (1 + \varepsilon u^p)^{q/2}$$

$$= \int_o^u \{1 + \varepsilon \frac{q}{2} \sigma^p + \frac{q}{2} (\frac{q}{2} - 1)\frac{(\varepsilon\sigma^p)^2}{2!} + \ldots\}d\sigma \qquad (4.17)$$

$$= u + \frac{q}{2} \frac{\varepsilon u^{p+1}}{p+1} + \frac{q}{2} (\frac{q}{2} - 1) \frac{\varepsilon^2}{2!} \frac{u^{2p+1}}{2p+1} + \ldots$$

Because of our bound $\varepsilon|u^p| < 1$, $(1+\varepsilon u^p)^{q/2}$ is an analytic function of u , and so is its integral. Thus the series (4.17) converges and has a non-zero derivative at $u = 0$. As a result, it can be inverted to give

$$u = \sum_{i=o}^{\infty} a_i(r+s)^i ,$$

for some bound δ, $|r+s| < \delta$. The first few terms are:

$$u = (r+s) - \frac{\varepsilon q}{2(p+1)} (r+s)^{p+1}$$

$$+ \varepsilon^2 [\frac{q^2}{4(p+1)} - \frac{q(q-2)}{8(2p+1)}](r+s)^{2p+1} + \ldots . \qquad (4.18)$$

In order to determine the region of convergence, we let $\mu = r+s$ and consider

$$\mu = \int_o^u (1 + \varepsilon\alpha^p)^{q/2} d\alpha \qquad (4.19)$$

so that

$$\frac{du}{d\mu} = \frac{1}{(1+\varepsilon u^p)^{q/2}} \qquad (4.20)$$

$$u(0) = 0 .$$

Now, consider the differential equation

$$\frac{du}{d\mu} = f(u)$$
$$u(0) = 0 \tag{4.21}$$

where $f(u)$ is analytic for $|u| < b$ and $|f(u)| < M$ there. It is well known, for example see Goursat [11], that the series solution for this equation is dominated by the series solution for

$$\frac{dU}{d\mu} = \frac{M}{1 - \frac{U}{b}}$$
$$U(0) = 0 \tag{4.22}$$

Further, the solution of (4.19) is given by

$$U - \frac{U^2}{2b} = M_\mu$$
$$U^2 - 2bU + 2bM_\mu = 0 \tag{4.23}$$
$$U = b[1 \pm \sqrt{1 - \frac{2M_\mu}{b}}]$$

which is analytic in μ for $|\mu| < \frac{b}{2M}$. In our case,

$$M = \max (1 + \varepsilon u^p)^{-\frac{q}{2}}$$
$$= \begin{cases} (1 - \varepsilon u_1^p)^{-\frac{q}{2}} & q > 0 \\ (1 + \varepsilon u_1^p)^{-\frac{q}{2}} & q < 0 \end{cases} \tag{4.24}$$

where $u_1 = \max u$, and $b \leq \frac{1}{\varepsilon^p}$. Thus (4.18) is convergent for

$$|r+s| < \frac{(1 - \varepsilon u_1^p)^{-\frac{q}{2}}}{2\varepsilon^p} \qquad q > 0 \ ,$$

$$|r+s| < \frac{(1 + \varepsilon u_1^p)^{-\frac{q}{2}}}{2\varepsilon^p} \qquad q < 0 \ . \tag{4.25}$$

Clearly, we may choose ε small enough to cover all values of $r+s$.

Again, as r and s are constant along lines ξ = const. and η = const., respectively, they have values from the initial line calulated as in

$$r(\sigma) = s(\sigma) = \frac{1}{2} \int_0^{f(\sigma)} (1 + \varepsilon\alpha^p)^{\frac{q}{2}} \, d\alpha$$

$$= \frac{1}{2} \left[f(\sigma) + \varepsilon \frac{q}{2} \frac{f(\sigma)^{p+1}}{p+1} + \ldots\right] \ . \tag{4.26}$$

Again r and s , and thus $u(\mu) = u(r+s)$ reach their maximum and minimum values where the initial function $u = f(\sigma)$ reaches its maximum and minimum. Thus, substituting the results from (4.26) into (4.12) and (4.18), that is, the values for $r(\xi)$, $s(\eta)$ into $v = s-r$ and $u = u(r+s)$, we obtain $u(\xi,\eta)$, $v(\xi,\eta)$ and $|u(\xi,\eta)| \leq \max f(\sigma) \leq \frac{1}{\varepsilon^p}$.

We next use the method of Fox and Lin, outlined in Chapter II, to obtain first terms of the expansion of the solution for x and t . Since we have $|\varepsilon u^p| < 1$, we can express

$$\Phi = (1 + \varepsilon u^p)^{\frac{q}{2}}$$

$$= 1 + \frac{q}{2} \varepsilon u^p + \frac{\frac{q}{2}(\frac{q}{2} - 1)}{2!} (\varepsilon u^p)^2 + \ldots \tag{4.27}$$

and from (4.18) we have

$$u = (r+s) - \frac{q}{2(p+1)} (r+s)^{p+1} + \ldots \quad , \tag{4.28}$$

so that

$$\Phi = 1 + \frac{q}{2} \varepsilon\{(r+s)^p - \frac{qp}{2(p+1)}(r+s)^{2p} + \ldots \}$$

$$+ \frac{q(q-2)}{8} \varepsilon^2 \{(r+s)^{2p} + \ldots \}$$

$$= 1 + \varepsilon \frac{q}{2} (r+s)^p + \varepsilon^2 [\frac{q(q-2)}{8} - \frac{q^2 p}{4(p+1)}](r+s)^{2p} + \ldots \quad . \tag{4.29}$$

Since

$$r(\sigma) = s(\sigma) = \frac{1}{2} \int_0^{f(\sigma)} (1 + \varepsilon\alpha^p)^{q/2} \, d\alpha$$

$$= \frac{1}{2} f(\sigma) + \frac{\varepsilon q \, f(\sigma)^{p+1}}{4(p+1)} + \ldots \quad , \tag{4.30}$$

where $f(x) = an\Pi \cos n\Pi x$, we have

$$r+s = \frac{1}{2}(f(\eta) + f(\xi)) + \frac{\varepsilon q}{4(p+1)}(f(\eta)^{p+1} + f(\xi)^{p+1}) + \ldots \quad . \tag{4.31}$$

Finally,

$$\Phi(\xi,\eta) = 1 + \frac{q}{2^{p+1}} (f(\xi)^{p+1} + f(\eta)^{p+1})$$

$$+ \varepsilon^2 \{[\frac{q(q-2)}{8} - \frac{q^2 p}{4(p+1)}](\frac{f(\xi) + f(\eta)}{2})^{2p} +$$

$$+ \frac{q^2 p}{8(p+1)} (\frac{f(\xi) + f(\eta)}{2})^{p+1} (f(\xi)^{p+1} + f(\eta)^{p+1})\}$$

$$+ 0(\varepsilon^3) \quad . \tag{4.32}$$

We now set $P = \Phi$ and $Q = -\Phi$ so that, from (2.31)

$$x^o(\xi,\eta) = \frac{\eta + \xi}{2} \quad ,$$
$$t^o(\xi,\eta) = \frac{\eta - \xi}{2} \quad , \tag{4.33}$$

and from (2.32)

$$t^{(1)} = -\frac{1}{2}\left[\int_\xi^\eta \frac{1}{2}\cdot\frac{q}{2}\left(\frac{f(\xi)+f(\sigma)}{2}\right)^p d\sigma + \int_\xi^\eta \frac{1}{2}\cdot\frac{q}{2}\left(\frac{f(\sigma)+f(\eta)}{2}\right)^p d\sigma\right]$$

$$= \frac{-q}{2^{p+3}} \sum_{k=o}^{p} \binom{p}{k} \left(f(\xi)^{p-k} + f(\eta)^{p-k}\right) \int_\xi^\eta f(\sigma)^k \, d\sigma \ , \tag{4.34}$$

$$x^{(1)} = \frac{q}{2^{p+3}} \sum_{k=o}^{p} \binom{p}{k} \left(f(\xi)^{p-k} - f(\eta)^{p-k}\right) \int_\xi^\eta f(\sigma)^k \, d\sigma \quad .$$

In particular, for $p = 1$, $q = 1$, $n = 1$, $f(\xi) = a\Pi \cos \Pi\xi$ we have:

$$t^{(1)}(\xi,\eta) = \frac{a\Pi}{16}(\eta-\xi)(\cos \Pi\xi - \cos \Pi\eta) + \frac{a}{8}(\sin \Pi\eta \sin \Pi\xi)$$
$$x^{(1)}(\xi,\eta) = \frac{a\Pi}{16}(\eta-\xi)(\cos \Pi\xi - \cos \Pi\eta) \quad , \tag{4.35}$$

and for $p = 2$, $q = 1$, $n = 1$ we have

$$\begin{aligned} t^{(1)}(\xi,\eta) = &-\frac{(a\Pi)^2}{32} (\eta-\xi)(\cos^2\Pi\xi + \cos^2\Pi\eta) \\ &-\frac{a^2\Pi}{16} (\cos \Pi\xi + \cos \Pi\eta)(\sin \Pi\eta - \sin \Pi\xi) \\ &-\frac{a^2\Pi}{32} \left[(\eta-\xi) + \frac{1}{2}(\sin 2\Pi\eta - \sin 2\Pi\xi)\right] \end{aligned} \tag{4.36}$$

$$\begin{aligned} x^{(1)}(\xi,\eta) = &\frac{(a\Pi)^2}{32} (\eta-\xi)(\cos^2\Pi\xi - \cos^2\Pi\eta) \\ &+\frac{a^2\Pi}{16} (\cos \Pi\xi - \cos \Pi\eta)(\sin \Pi\eta - \sin \Pi\xi) \quad . \end{aligned}$$

In general, when we set $f(x)' = (an\Pi) \cos n\Pi x$, and observe that

$$G^k(\sigma) = \int \frac{1}{n\Pi} \cos^k \sigma d\sigma$$

$$= \frac{1}{n\Pi} \left\{ \frac{1}{k} \cos^{k-1}\sigma \sin\sigma + \frac{k-1}{k} \frac{1}{k-2} \cos^{k-3}\sigma \sin\sigma \right. \tag{4.37}$$

$$\left. + \dots + \left\{ \begin{array}{ll} \frac{k-1}{k} \dots \frac{3}{4} \left(\frac{\sigma}{2} + \frac{1}{2} \cdot \sin\sigma \cos\sigma\right) & k \text{ even} \\ \frac{k-1}{k} \dots \frac{2}{3} \sin\sigma & k \text{ odd} \end{array} \right\} \right\} + c \ .$$

Then we obtain:

$$t^{(1)}(\xi,\eta) = \frac{-q}{2^{p+3}}(an\Pi)^p \sum_{k=o}^{p} \left[\binom{p}{k}(\cos^{p-k} n\Pi\xi + \cos^{p-k} n\Pi\eta)\right.$$

$$\left. \cdot (G^k(n\Pi\eta) - G^k(n\Pi\xi))\right] \ . \tag{4.38}$$

$$x^{(1)}(\xi,\eta) = \frac{q}{2^{p+3}}(an\Pi)^p \sum_{k=o}^{p} \left[\binom{p}{k}(\cos^{p-k} n\Pi\xi - \cos^{p-k} n\Pi\eta)\right.$$

$$\left. \cdot (G^k(n\Pi\eta) - G^k(n\Pi\xi))\right] \ . \tag{4.39}$$

Taking derivatives of these, one could calculate further terms of the solution series according to (2.32).

Under our assumption that we deal with a region where $J \neq 0$, we attempt to find $\xi(x,t)$, $\eta(x,t)$. We seek expressions of the form

$$\begin{aligned} \xi &= \xi^o(x,t) + \varepsilon\, \xi^{(1)}(x,t) \\ \eta &= \eta^o(x,t) + \varepsilon\, \eta^{(1)}(x,t) \ . \end{aligned} \tag{4.40}$$

The terms ξ^o , η^o are the unperturbed linear expressions $x - t$, $x + t$. From expressions of x and t ,

$$t = \frac{\eta - \xi}{2} + \varepsilon\, t^{(1)}$$

$$x = \frac{\eta + \xi}{2} + \varepsilon\, x^{(1)} \quad , \tag{4.41}$$

we obtain

$$\xi = x - t + \varepsilon(x^{(1)} - t^{(1)}) \quad ,$$

$$x = x + t + \varepsilon(x^{(1)} + t^{(1)}) \quad .$$

for $p = 1$, $q = 1$, $n = 1$, from (4.35), we have

$$\xi = x-t + \varepsilon[\frac{a\Pi}{8} (\eta-\xi) \cos \Pi\eta + \frac{a}{8} (\cos \Pi\eta - \sin \Pi\xi)]$$

$$= x-t + \varepsilon[\frac{a\Pi}{4} t \cos \Pi(x+t) + \frac{a}{4} \cos \Pi x \sin \Pi t] \quad , \tag{4.42}$$

$$\eta = x+t - \varepsilon[\frac{a\Pi}{4} t \cos \Pi\, (x-t) + \frac{a}{4} \sin \Pi x \cos \Pi t] \quad .$$

For t small, the series of which these are the first terms surely give a good approximation where ε is small. But as $t \to t_b$, higher order terms may become significant.

Similarly for $p = 2$, $q = 1$, $n = 1$, from (4.36), we have

$$\xi = x - t + \varepsilon \frac{a^2\pi}{8} t(\pi \cos^2 \pi(x-t) + \frac{1}{2})$$

$$+ \varepsilon \frac{a^2\pi}{8} \cos \pi(x-t) \cos \pi x \sin \pi t + \frac{1}{4} \cos 2\pi x \sin 2\pi t \tag{4.43}$$

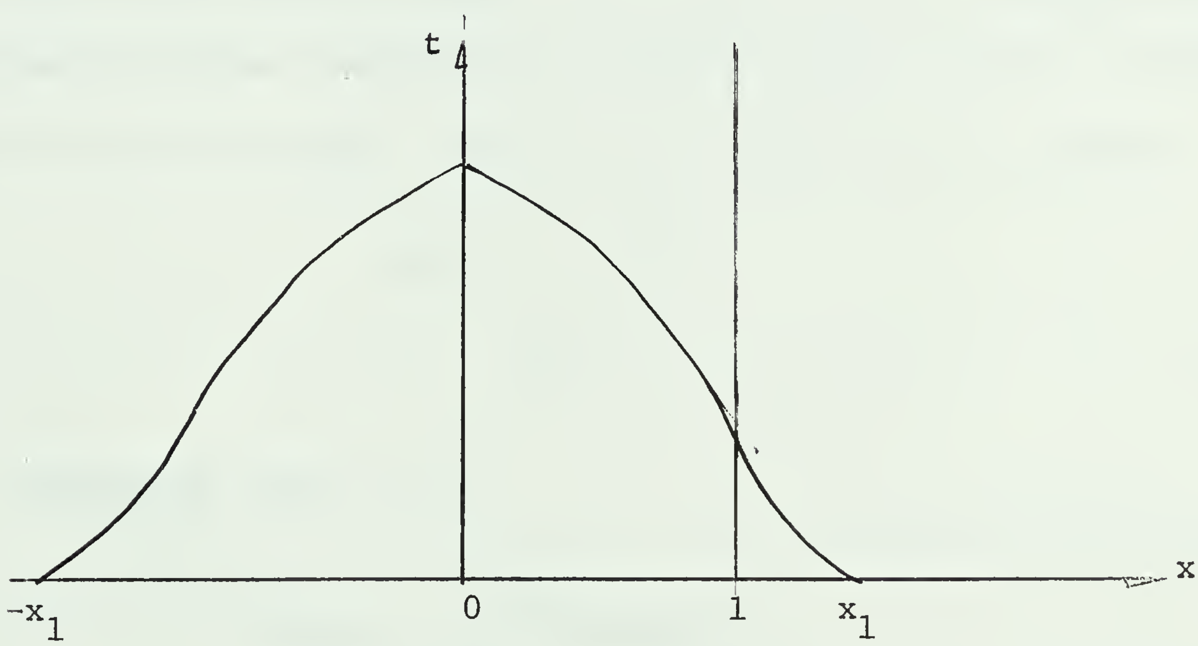
$$\eta = x + t - \varepsilon \frac{a^2\pi^2}{8} t \cos^2 \pi(x+t)$$

$$- \varepsilon \frac{a^2\pi}{8} \cos \pi(x+t) \cos \pi x \sin \pi t \quad .$$

Thus, combining values of $u(r,s)$, $v(r,s)$, values of $r(\xi)$, $s(\eta)$, and those of $\xi(x,t)$, $\eta(x,t)$, one can arrive at $u,v = (U(x,t), V(x,t))$ form which, $Y(x,t)$ could be calculated.

We have considered the problem in the half plane, but the solution also satisfies the end conditions of the fixed end problem:

$$Y(0,t) = Y(1,t) = 0 \quad . \tag{4.44}$$



CHARACTERISTIC LINES INTERSECTING IN $x = 0$.

Figure X

Because $U(x,0) = U(-x,0)$ and $V(x,0) = 0$, from (4.10) we have $r(\sigma) = s(\sigma) = s(-\sigma)$. Thus, from (4.11)

$$\begin{aligned} v(\sigma,-\sigma) &= r(\sigma) - s(-\sigma) \\ &= 0 \end{aligned} \tag{4.45}$$

and

$$\begin{aligned} u(\sigma,-\sigma) &= u(r(\sigma) + s(-\sigma)) \\ &= u(r(-\sigma) + s(\sigma)) \\ &= u(-\sigma,\sigma) \end{aligned} \tag{4.46}$$

and further

$$\Phi(\sigma,-\sigma) = \Phi(-\sigma,\sigma) \quad . \tag{4.47}$$

From this we see that the slopes, $P = \Phi$ and $Q = -\Phi$ of the characteristic lines, and the lines themselves are symmetric about $x = 0$. It then follows that for any point $x(\xi,\eta) = 0$, we have $\xi = -\eta$ and so

$$\begin{aligned} V(0,t) &= v(-\eta,\eta) \\ &= 0 \quad . \end{aligned} \tag{4.48}$$

Thus, integrating along $x = 0$

$$\begin{aligned} Y(0,t_1) &= \int_o^{t_1} V(0,t)dt \\ &= 0 \quad . \end{aligned} \tag{4.49}$$

A similar analysis can be performed for $x = 1$ as initial data is also

symmetric about $x = 1$.

We can now indicate possible breakdown times, using the approximations we have to t . We do not, in fact, prove that breakdown occurs, but that only under the assumption that the neglected terms are small, we calculate points where t_ξ or t_η are zero. From (4.34), we have that

$$t_\xi^{(1)}(\xi,\eta) = \frac{q}{2^{p+3}} \sum_{k=o}^{p} \binom{p}{k} [f(\xi)^{p-k} + f(\eta)^{p-k}]f(\xi)^k$$
$$- \frac{q}{2^{p+3}} \sum_{k=o}^{p} \binom{p}{k} (p-k)f(\xi)^{p-k-1} f'(\xi) \int_\xi^\eta f(\sigma)^k d\sigma \tag{4.50}$$

and

$$t_\eta^{(1)}(\xi,\eta) = -t_\xi^{(1)}(\eta,\xi) \quad . \tag{4.51}$$

We observe that $\varepsilon t_\xi^{(1)}$ is $0(1)$ only in those terms for which

$$\int_\xi^\eta f(\sigma)^k d\sigma = (\eta-\xi)h(\xi,\eta) \quad , \tag{4.52}$$

in which case h is constant. In this case, $\varepsilon(\eta-\xi)$ is $0(1)$ when $(\eta-\xi) = \frac{1}{\varepsilon}$.

Setting $f(x) = an\Pi \cos n\Pi x$ in (4.50) and using the integration result for $\int_\xi^\eta \cos^k(\sigma)d\sigma$, (4.38) we obtain:

$$t = -\frac{1}{2} + \frac{\varepsilon\ qp}{2^{p+3}} \sum_{\substack{k=o \\ k\ \text{even}}}^{p-1} \binom{p-1}{k} (a\pi n)^p \cos^{p-2k-1} n\Pi\xi \ \sin n\Pi\xi$$

$$\cdot \prod_{m=1}^{k/2} \frac{k-2m+1}{k-2m+2} \cdot (\eta-\xi) + 0(\varepsilon) \tag{4.53}$$

$$= -\frac{1}{2} + \varepsilon(\eta-\xi) \frac{pq\Pi n(a\Pi n)^p}{2^{p+3}} \sum_{k=o}^{INT(\frac{p-1}{2})} \binom{p-1}{2k} \frac{2k!}{(k!)^2\ 2^{2k}}$$

$$\cdot \cos^{p-2k-1} n\Pi\xi \ \sin n\Pi\xi + 0(\varepsilon) \quad .$$

Also $t_\eta(\xi,\eta) = -t_\xi(\eta,\xi)$ to the order of ε . We approximate the singular curve by the curve $J = 0(\varepsilon)$ as $\varepsilon \to 0$, in keeping with our assumption that the sum $\sum_{k=2}^{\infty} \varepsilon^k t^k(\xi,\eta)$ is negligible. So the singular curves are approximated by

$$t_\xi = 0(\varepsilon) \text{ as } \varepsilon \to 0 \text{ and } t_\eta = 0(\varepsilon) \text{ as } \varepsilon \to 0 \ .$$

For $q = 1$, $p = 1$, we have

$$\eta = \xi + 8[\varepsilon(\Pi n)^2\ a \sin n\Pi\xi]^{-1} \quad . \tag{4.54}$$

For $q = 1$, $p = 2$,

$$\eta = \xi + 16[\varepsilon(\Pi n)^3\ a^2 \sin 2n\Pi\xi]^{-1} \quad . \tag{4.55}$$

In the general case the curves:

$$-\frac{1}{2} + \frac{pq\Pi n(a\Pi n)^p}{2^{p+3}} \varepsilon(\eta-\xi)\ M(\xi) = 0 \quad , \tag{4.56}$$

where

$$M(\xi) = \sum_{k=o}^{INT(\frac{p-1}{2})} \binom{p-1}{2k} \frac{2k!}{(k!)^2 2^{2k}} \cos^{p-2k-1} n\Pi\xi \sin n\Pi\xi$$

trace out the approximation to the set of points in the (ξ, η) plane where $J = 0$. If we choose ξ_1 for which $M(\xi)$ has its maximum value and we choose η_1 such that

$$\eta_1 - \xi_1 = 2^{p+2}[\varepsilon pq(\Pi n)^{p+1} a^p \max M(\xi)]^{-1} , \tag{4.57}$$

we have an approximation to the minimum time for which $J = 0$.

Increasing or decreasing η slightly from η_1, the actual zero of J and t_ξ should occur, as $(\eta - \xi_1)$ varies.

Similarly we have $J \doteq 0$ for

$$\eta - \xi = 2^{p+2}[\varepsilon pq(\Pi n)^{p+1} a^p \max M(\eta)]^{-1} \tag{4.58}$$

when $t_\eta = 0(\varepsilon)$.

We can use the linear result for t, $t = \frac{\eta - \xi}{2}$ to obtain an approximation for the initial time, t_b for which breakdown occurs.

$$t_b = 2^{p+1}[\varepsilon pq(\Pi n)^{p+1} a^p \max M(\eta)]^{-1} . \tag{4.59}$$

In this way, we arrive at a prediction of initial breakdown time for the two cases studied in the F.P.U. report.

For $p = 1$, $q = 1$, $n = 1$, we have

$$M(\xi) = \sin \Pi\xi$$

$$\max M(\xi) = M(1/2) = 1 \tag{4.60}$$

$$t_b = \frac{4}{\varepsilon a \Pi^2} \quad .$$

This result is in agreement with the papers which dealt with this problem. For $p = 2$, $q = 1$, $n = 1$, the other F.P.U. problem,

$$M(\xi) = \sin 2\Pi\xi$$

$$\max M(\xi) = M(1/4) = 1 \tag{4.61}$$

$$t_b = \frac{8}{\varepsilon a^2 \Pi^3} \quad .$$

One notes further from (4.59), that initial breakdown time t_b decreases as the parameters p, q, ε, n and a are decreased.

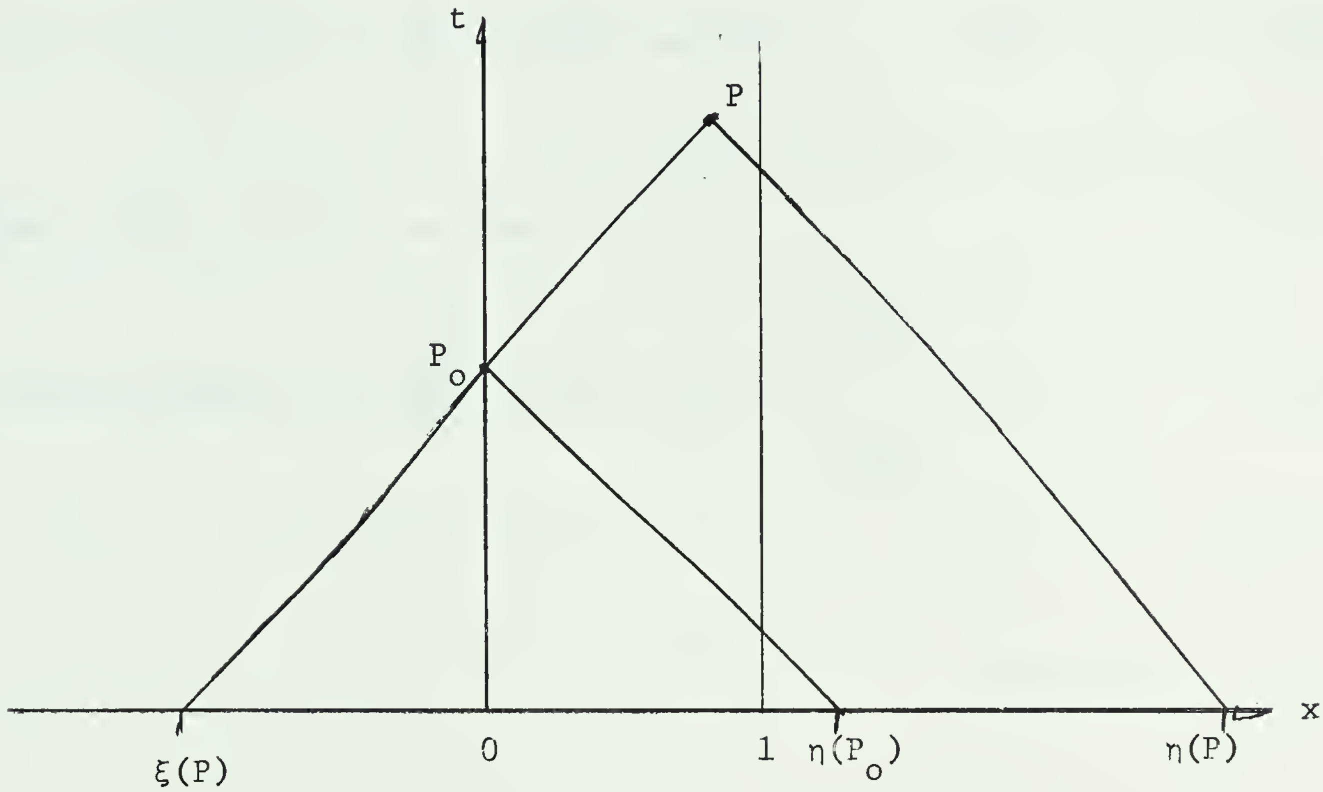
Finally, we outline the calculation of $Y(P)$ at a point $P(x,t)$. We calculate $\xi(P)$ and $\eta(P)$ from (4.43) and find the point P_o where the line $\xi = \xi(P)$ intersects $x = 0$ or $t = 0$. When $\xi(P) < 0$, $\eta(P_o)$ is determined from (4.42) such that $x(\xi(P),\eta(P_o)) = 0$, and we have $Y(P_o) = 0$. For $\xi(P) \geq 0$, P_o is on $t = 0$ and $Y(P_o) = a \sin \Pi\xi(P)$. We calculate for $p = 1$, $q = 1$, $n = 1$.

We consider the first term of these series expressions:

$$u(\xi,\eta) = \frac{a\Pi}{2} [\cos \Pi\eta + \cos \Pi\xi]$$

$$v(\xi,\eta) = \frac{a\Pi}{2} [\cos \Pi\eta - \cos \Pi\xi] \tag{4.62}$$

$$t_\eta(\xi,\eta) = \frac{1}{2} + \varepsilon\frac{a\Pi^2}{16}(\eta-\xi)\sin\Pi\eta$$
$$x_\eta(\xi,\eta) = \frac{1}{2} + \varepsilon\frac{a\Pi^2}{16}(\eta-\xi)\sin\Pi\eta \tag{4.63}$$



THE (x,t) PLANE

Figure XI

We assume that $\xi(P) < 0$, and we calculate:

$$\begin{aligned} Y(P) &= \int_{\eta(P_o)}^{\eta(P)} Y_\eta \, d\eta \\ &= \int_{\eta(P_o)}^{\eta(P)} [ux_\eta + vt_\eta] d\eta \\ &= \int_{\eta(P_o)}^{\eta(P)} [2\cos\Pi\delta + \varepsilon\frac{a\Pi^2}{16}(\delta-\xi)\sin\Pi\delta] d\delta \quad . \end{aligned} \tag{4.64}$$

For small t and small ε, the second term is insignificant; for $t = O(\frac{1}{\varepsilon})$, $\eta-\xi$ can be considered constant over the interval $(\eta(P_o),\eta(P))$, so for $Y(P_o)$ on $x = 0$ we have

$$Y(P) = \frac{2}{\Pi} \sin \Pi\delta - \varepsilon \frac{a}{8} (\delta-\xi(P)) \cos \Pi\delta \Big|_{\eta(P_o)}^{\eta(P)} + O(\varepsilon) \quad . \tag{4.65}$$

Similarly, for $p = 2$, we have

$$Y(P) = \frac{2}{\Pi} \sin \Pi\delta - \varepsilon \frac{a\Pi^2}{16} (\delta-\xi(P)) \cos^2 \Pi\delta \Big|_{\eta(P_o)}^{\eta(P)} + O(\varepsilon) \quad . \tag{4.66}$$

CHAPTER V

THE MOVING THREADLINE

We now attempt to apply the preceding methods to the problem of determining the motion of the "moving threadline". We shall investigate two models for its equation of motion; for details of the derivation of these see Zaiser [18],

$$\frac{\alpha^2}{4} Y_{xx} + \alpha Y_{xt} + Y_{tt} = (1 + Y_x^2)^{-1} Y_{xx} \,, \qquad \text{Case I} \,, \tag{5.1}$$

$$\frac{\alpha^2}{4} (1+Y_x^2)^{-1} Y_{xx} + \alpha(1+Y_x^2)^{-\frac{1}{2}} Y_{xt} + Y_{tt} = (1+Y_x^2)^{-1} \,, \qquad \text{Case II} \,. \tag{5.2}$$

We shall impose smooth perturbations in the boundary conditions, of the form,

$$\left.\begin{aligned} Y(x,0) &= \varepsilon \sin \Pi x \\ Y_t(x,0) &= 0 \,, \end{aligned}\right\} \qquad 0 \leq x \leq 1 \,, \tag{5.3}$$

$$Y(0,t) = Y(1,t) = 0 \,, \qquad t > 0 \,, \tag{5.4}$$

and for reasons to be shown later we take $0 < \alpha < 2$.

We write the equations as a first order system by setting $Y_x = u$, $Y_t = v$, and thus obtain:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 \,, & 1 \\ (1+u^2)^{-1} - \frac{\alpha^2}{4} (1+u^2)^k \,, & -\alpha(1+u^2)^{\frac{k}{2}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x \tag{5.5}$$

where $k = 0, -1$ in Cases I and II respectively, with initial conditions

$$\begin{aligned} u &= U(x,0) = \varepsilon\Pi \cos \Pi x , \\ v &= V(x,0) = 0 . \end{aligned} \qquad 0 \leq x \leq 1 , \tag{5.6}$$

The solution for the "infinite" threadline, i.e. ignoring the equivalent to equation (5.4), can be obtained in a manner similar to the previous problem. But the end conditions for the finite interval cannot be as easily satisfied as they were in the string problem.

We look at the corresponding linear problem in order to establish initial conditions for the infinite string which will approximate those needed to provide for the zero end conditions in the nonlinear problems. We consider the equation

$$Y_{tt} + \alpha Y_{xt} + (\frac{\alpha^2}{4} - 1) Y_{xx} = 0 , \qquad \begin{matrix} t > 0 , \\ 0 < x < 1 , \end{matrix} \tag{5.7}$$

with

$$Y(0,t) = g(x) , \qquad 0 \leq x \leq 1 , \tag{5.8}$$

where $g(0) = g(1) = 0$,

$$Y_t(x,0) = 0 ,$$

$$Y(0,t) = Y(1,t) = 0 , \qquad t \geq 0 . \tag{5.9}$$

Setting $u = Y_x$ and $v = Y_t$, as in the previous problem we obtain:

$$\begin{aligned} v_\xi + (\frac{\alpha}{2} + 1) u_\xi &= 0 \qquad u(\sigma,\sigma) = f(\sigma) , \\ v_\eta + (\frac{\alpha}{2} - 1) u_\eta &= 0 \qquad v(\sigma,\sigma) = 0 , \end{aligned} \tag{5.10}$$

$$x_\xi - (\frac{\alpha}{2} - 1)\, t_\xi = 0 \qquad x(\sigma,\sigma) = \sigma \quad ,$$
$$x_\eta - (\frac{\alpha}{2} + 1)\, t_\eta = 0 \qquad t(\sigma,\sigma) = 0 \quad , \tag{5.11}$$

where $f(\sigma)$ will be defined later, and is assumed to be sufficiently smooth. We choose Riemann invariants r and s such that:

$$s_v = \frac{1}{2} \qquad r_v = -\frac{1}{2} \quad ,$$
$$s_u = \frac{1}{2}(\frac{\alpha}{2} + 1) \qquad r_u = -\frac{1}{2}(\frac{\alpha}{2} - 1) \quad . \tag{5.12}$$

Integrating these we obtain:

$$r = r(\xi) = \frac{1}{2}(1 - \frac{\alpha}{2})u - \frac{v}{2} \quad ,$$
$$s = s(\eta) = \frac{1}{2}(1 + \frac{\alpha}{2})u + \frac{v}{2} \quad , \tag{5.13}$$

so that, in the usual way,

$$u(\xi,\eta) = r(\xi) + s(\eta) \quad ,$$
$$v(\xi,\eta) = (1 - \frac{\alpha}{2})s(\eta) - (1 + \frac{\alpha}{2})r(\xi) \quad , \tag{5.14}$$

and we have, from the initial line

$$r(\xi) = \frac{1}{2}(1 - \frac{\alpha}{2})f(\xi) \quad ,$$
$$s(\eta) = \frac{1}{2}(1 + \frac{\alpha}{2})f(\eta) \quad . \tag{5.15}$$

Again from (5.11) we have

$$x = \frac{1}{2}(1 + \frac{\alpha}{2})\eta + \frac{1}{2}(1 - \frac{\alpha}{2})\xi \quad ,$$
$$t = \frac{1}{2}(\eta - \xi) \quad ; \tag{5.16}$$

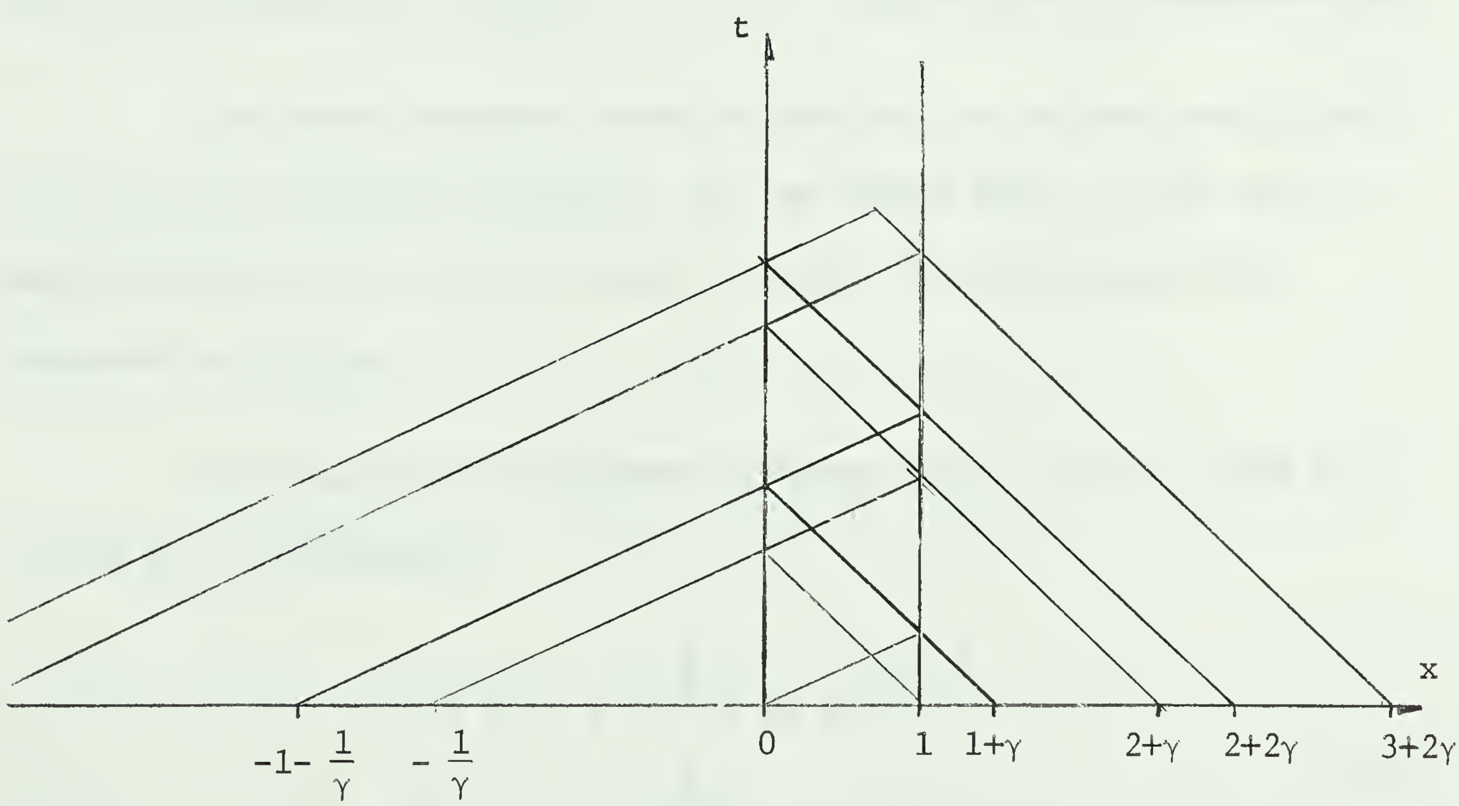
and thus

$$\eta = x + (1 - \frac{\alpha}{2})t \quad ,$$
$$\xi = x - (1 + \frac{\alpha}{2})t \quad . \tag{5.17}$$

Now, in order to satisfy the end conditions, the initial conditions are extended to $(-\infty,\infty)$ in the following fashion. Let $h(x) = \frac{dg(x)}{dx}$, and let

$$\begin{aligned} h(x) &= h(x) \quad , & 0 \le x \le 1 \ , \\ &= h(\frac{\gamma+1-x}{\gamma}) \quad , & 1 \le x \le 1+\gamma \\ &= h(-\gamma x) \quad , & -\frac{1}{\gamma} \le x \le 0 \quad , \end{aligned} \tag{5.18}$$

where $\gamma = (1 - \frac{\alpha}{2})/(1 + \frac{\alpha}{2})$. Then the initial conditions on $[0,1+\gamma]$ are extended periodically to the right and those of $[-\frac{1}{\gamma}, 1]$, periodically to the left.



LINEAR CHARACTERISTIC LINES FOR MOVING THREADLINE

Figure XII

With this extension we see that, for example, at the point $(0, \frac{1}{1-\frac{\alpha}{2}})$ we have $\xi = \frac{-1}{\gamma}$, $\eta = 1$; for which

$$r(-\frac{1}{\gamma}) = \frac{1}{2}(1 - \frac{\alpha}{2})f(-\frac{1}{\gamma}) = \frac{1}{2}(1 - \frac{\alpha}{2})h(1) \quad ,$$
$$s(1) = \frac{1}{2}(1 + \frac{\alpha}{2})f(1) = \frac{1}{2}(1 + \frac{\alpha}{2})h(1) \quad , \tag{5.20}$$

and thus

$$\begin{aligned} v &= \frac{1}{2}(1 - \frac{\alpha}{2})(1 + \frac{\alpha}{2})h(1) - \frac{1}{2}(1 + \frac{\alpha}{2})(1 - \frac{\alpha}{2})h(1) \ , \\ &= 0 \quad . \end{aligned} \tag{5.21}$$

If $v = 0$ at every point on $x = 0$ and $x = 1$, then we have $Y = \int_0^t v dt = 0$ there.

Physically, it is clear that $\alpha < 2$. It is also evident from the characteristics that for $\alpha \geq 2$ the problem is overdetermined.

The above extension does not provide for the end conditions, (5.9), in the nonlinear problems. But we assume that, in the case of small perturbations from the linear problem, it approximates the required conditions.

We consider the nonlinear problems (5.6). We set $h(x) = \varepsilon\Pi \cos \Pi x$, and define

$$\begin{aligned} \phi^+ &= \frac{\alpha}{2}(1 + u^2)^{\frac{k}{2}} + (1 + u^2)^{-\frac{1}{2}} \quad , \\ \phi^- &= \frac{\alpha}{2}(1 + u^2)^{\frac{k}{2}} - (1 + u^2)^{-\frac{1}{2}} \quad , \\ & \qquad k = 0, -1 \quad . \end{aligned} \tag{5.22}$$

Then, the eigenvalues of the matrix in (5.5) are $-\Phi^-$, $-\Phi^+$, and left eigenvectors are $(\Phi^+,1)$ and $(\Phi^-,1)$ respectively. A brief calculation, as in previous chapters, provides us with the equations in characteristic coordinates,

$$\begin{aligned} v_\xi + \Phi^+ u_\xi &= 0 \quad , \\ v_\eta + \Phi^- u_\eta &= 0 \quad , \end{aligned} \tag{5.23}$$

$$\begin{aligned} x_\xi - \Phi^- t_\xi &= 0 \quad , \\ x_\eta - \Phi^+ t_\eta &= 0 \quad . \end{aligned} \tag{5.24}$$

The initial conditions are then given by

$$\begin{aligned} u(\sigma,\sigma) &= f(\sigma) \quad , \\ v(\sigma,\sigma) &= 0 \quad , \end{aligned} \tag{5.25}$$

$$\begin{aligned} x(\sigma,\sigma) &= \sigma \quad , \\ t(\sigma,\sigma) &= 0 \quad . \end{aligned} \tag{5.26}$$

We choose the Riemann invariants,

$$\begin{aligned} r = r(\xi) &= -\frac{v}{2} - \frac{1}{2}\int_o^u \left[\frac{\alpha}{2}(1+\gamma^2)^{\frac{k}{2}} - (1+\gamma^2)^{-\frac{1}{2}}\right]d\gamma \quad , \\ s = s(\eta) &= \frac{v}{2} + \frac{1}{2}\int_o^u \left[\frac{\alpha}{2}(1+\gamma^2)^{\frac{k}{2}} + (1+\gamma^2)^{-\frac{1}{2}}\right]d\gamma \quad , \end{aligned} \tag{5.27}$$

so that we have

$$\begin{aligned} r + s &= \int_o^u (1+\gamma^2)^{-\frac{1}{2}} d\gamma \\ &= \sinh^{-1} u \quad . \end{aligned} \tag{5.27}$$

$$s - r = v + \frac{\alpha}{2} \int_o^u (1+\gamma^2)^{\frac{k}{2}} \, d\gamma$$

$$= \begin{cases} v + \frac{\alpha}{2} u \quad , & k = 0 \; , \\ v + \frac{\alpha}{2} \sinh^{-1} u \quad , & k = 1 \; . \end{cases} \tag{5.28}$$

Since $\sinh^{-1} u$ is a monotonic increasing function of u, we may invert the equation so that:

$$u = \sinh (r+s)$$

$$v = \begin{cases} s - r - \frac{\alpha}{2} \sinh (r+s) \; , & k = 0 \; , \\ s - r - \frac{\alpha}{2} (r+s) \; , & k = 1 \; . \end{cases} \tag{5.29}$$

Again, because r and s are constant along ξ = const. and η = const. respectively, we find from the initial conditions that:

$$\begin{aligned} r(\xi) &= -\frac{\alpha}{4} f(\xi) + \frac{1}{2} \sinh^{-1} f(\xi) \; , \\ s(\eta) &= \frac{\alpha}{4} f(\eta) + \frac{1}{2} \sinh^{-1} f(\eta) \; , \quad \text{if} \quad k = 0 \; , \end{aligned} \tag{5.30}$$

and

$$\begin{aligned} r(\xi) &= \frac{1}{2} \left(1 - \frac{\alpha}{2}\right) \sinh^{-1} f(\xi) \; , \\ s(\eta) &= \frac{1}{2} \left(1 + \frac{\alpha}{2}\right) \sinh^{-1} f(\eta) \; , \quad \text{if} \quad k = -1 \; . \end{aligned} \tag{5.31}$$

Thus, the solution to the first part of the problem can be expressed in the form:

$$u(\xi,\eta) = \sinh\left[\frac{\alpha}{4}(f(\eta)-f(\xi)) + \frac{1}{2}(\sinh^{-1}f(\xi) + \sinh^{-1}f(\eta))\right]$$

$$v(\xi,\eta) = \frac{\alpha}{4}(f(\xi)+f(\eta)) - \frac{1}{2}(\sinh^{-1}f(\eta) - \sinh^{-1}f(\xi)) - \frac{\alpha}{2}u(\xi,\eta) \qquad (5.32)$$

for $k = 0$.

and

$$u(\xi,\eta) = \sinh\left[\frac{1}{2}(1-\frac{\alpha}{2})\sinh^{-1}f(\xi) + \frac{1}{2}(1+\frac{\alpha}{2})\sinh^{-1}f(\eta)\right]$$

$$v(\xi,\eta) = \frac{1}{2}(1-\frac{\alpha}{2})(1+\frac{\alpha}{2})[\sinh^{-1}f(\eta)-\sinh^{-1}f(\xi)] \quad \text{for} \quad k = -1 . \qquad (5.33)$$

We now turn to consideration to the second system of equations

$$x_\xi - \Phi^- t_\xi = 0 \quad ,$$

$$x_\eta - \Phi^+ t_\eta = 0 \quad , \qquad (5.34)$$

$$x(\sigma,\sigma) = \sigma \qquad t(\sigma,\sigma) = 0 \quad . \qquad (5.35)$$

Firstly we note that $(1+u^2)^{\frac{1}{2}} = \cosh(r+s)$ so that

$$\Phi^{\pm} = \frac{\alpha}{2}[\cosh(r+s)]^k \pm [\cosh(r+s)]^{-1} \quad , \qquad (5.36)$$

with

$$r(\xi)+s(\eta) = \frac{\alpha}{4}[f(\eta)-f(\xi)] + \frac{1}{2}[\sinh^{-1}f(\xi)+\sinh^{-1}f(\eta)] \quad , \quad \text{if} \quad k = 0 ,$$

and (5.37)

$$r(\xi)+s(\eta) = \frac{1}{2}(1-\frac{\alpha}{2})\sinh^{-1}f(\xi) + \frac{1}{2}(1+\frac{\alpha}{2})\sinh^{-1}f(\eta) \quad , \quad \text{if} \quad k = 1 .$$

In order to express Φ^+, Φ^- in series, we set $f(x) = \varepsilon m(x)$, and note that if we maintain $\varepsilon < \frac{1}{4}$, so that $|r+s| < \frac{\Pi}{2}$, then

$\cosh (r+s)$ is an analytic function of $(r+s)$ and the series for $\Phi^{\pm}$ is convergent.

$$r(\xi) + s(\eta) = \frac{\varepsilon}{2}(1 + \frac{\alpha}{2})m(\eta) + \frac{\varepsilon}{2}(1 - \frac{\alpha}{2})m(\xi) + O(\varepsilon^3) \quad , \qquad (5.38)$$

$$[\cosh (r+s)]^{-1} = 1 - \frac{\varepsilon^2}{8}\,[(1 + \frac{\alpha}{2})m(\eta)+(1 - \frac{\alpha}{2})m(\xi)]^2 + O(\varepsilon^4). \qquad (5.39)$$

Thus we have

$$\begin{aligned} \Phi^{\pm} &= \frac{\alpha}{2} \pm (1 - \frac{\varepsilon^2}{8}\,C(\xi,\eta) + O(\varepsilon^4)) \quad \text{if} \quad k = 0 \quad , \\ \Phi^{\pm} &= (\frac{\alpha}{2} \pm 1)(1 - \frac{\varepsilon^2}{8}\,C(\xi,\eta) + O(\varepsilon^4)) \ , \quad \text{if} \quad k = -1 \quad , \end{aligned} \qquad (5.40)$$

where we define

$$C(\xi,\eta) = [(1 + \frac{\alpha}{2})m(\eta) + (1 - \frac{\alpha}{2})m(\xi)]^2 \quad . \qquad (5.41)$$

Proceeding as before with $P = \Phi^{\pm}$ and $Q = \Phi^{-}$, we integrate (5.34) to find

$$\begin{aligned} x-\eta &= (\frac{\alpha}{2} - 1)t(\xi,\eta) + \frac{\varepsilon^2}{8}(1 - \frac{\alpha}{2})^{-k} \int_{\eta}^{\xi} C(\sigma,\eta)t_{\sigma}(\sigma,\eta)d\sigma + \ldots \\ x-\xi &= (\frac{\alpha}{2} + 1)t(\xi,\sigma) - \frac{\varepsilon^2}{8}(\frac{\alpha}{2} + 1)^{-k} \int_{\xi}^{\eta} C(\xi,\sigma)t_{\sigma}(\xi,\sigma)d\sigma + \ldots \ , \\ & \qquad k = 0,-1 \ , \end{aligned} \qquad (5.42)$$

and, on subtracting, we have,

$$t(\xi,\eta) = \frac{\eta-\xi}{2} - \frac{\varepsilon^2}{16}\int_{\xi}^{\eta}[(1 - \frac{\alpha}{2})^{-k}C(\sigma,\eta)t_{\sigma}(\sigma,\eta)-(\frac{\alpha}{2} + 1)^{-k}C(\xi,\sigma)t_{\sigma}(\xi,\sigma)]d\sigma$$
$$+ \ldots \ , \quad k = 0,-1 \ . \qquad (5.43)$$

We look for solutions of the form:

$$t(\xi,\eta) = \sum_{i=o}^{\infty} \varepsilon^{2i}\, t^{(i)}(\xi,\eta) \quad .$$

Taking the first two terms of this series we obtain

$$t(\xi,\eta) = \frac{\eta-\xi}{2} + \frac{\varepsilon^2}{32}\int_{\xi}^{\eta} [(\tfrac{\alpha}{2}+1)^{-k}C(\xi,\sigma)+(1-\tfrac{\alpha}{2})^{-k}C(\sigma,\eta)]d\sigma \ , \quad (5.44)$$

$$x(\xi,\eta) = \frac{\eta+\xi}{2} + \frac{\alpha}{2}\, t(\xi,\eta) + \frac{\varepsilon^2}{32}\int_{\xi}^{\eta}[(1-\tfrac{\alpha}{2})^{-k}C(\sigma,\eta)-(\tfrac{\alpha}{2}+1)^{-k}C(\xi\ \sigma)]d\sigma, (5.45)$$

and so we may write

$$x(\xi,\eta) = \frac{1}{2}(1+\frac{\alpha}{2})\eta + \frac{1}{2}(1-\frac{\alpha}{2})\xi$$

$$+ \frac{\varepsilon^2}{32}\int_{\xi}^{\eta} [(\tfrac{\alpha}{2}-1)(\tfrac{\alpha}{2}+1)^{-k}C(\xi,\sigma)+(\tfrac{\alpha}{2}+1)(1-\tfrac{\alpha}{2})^{-k}C(\sigma,\eta)]d\sigma$$

$$k = 0,1 \quad . \qquad (5.46)$$

If the Jacobean $J = \frac{\partial(x,t)}{\partial(\xi,\eta)} \neq 0$, we may solve for $\xi(x,t)$, $\eta(x,t)$. Let us seek expressions of the form

$$\begin{aligned} \xi(x,t) &= \xi_o(x,t) + \varepsilon^2\xi_1(x,t) + \ldots \\ \eta(x,t) &= \eta_o(x,t) + \varepsilon^2\eta_1(x,t) + \ldots \quad . \end{aligned} \qquad (5.47)$$

We then find $\xi_o = x - (1+\frac{\alpha}{2})t$, $\eta_o = x + (1-\frac{\alpha}{2})t$. Combining (5.44) and (5.45) we have

$$\begin{aligned} \xi &= x-(1+\tfrac{\alpha}{2})t-\varepsilon^2[x^{(1)}(\xi,\eta)-(1+\tfrac{\alpha}{2})t^{(1)}(\xi,\eta)]+\ldots \\ \eta &= x+(1-\tfrac{\alpha}{2})t+\varepsilon^2[x^{(1)}(\xi,\eta)+(1-\tfrac{\alpha}{2})t^{(1)}(\xi,\eta)]+\ldots \quad . \end{aligned} \qquad (5.48)$$

From this, we have

$$\xi = \xi(x,t) = x-(1+\frac{\alpha}{2})t + \frac{\varepsilon^2}{16}\int_{x-(1+\frac{\alpha}{2})t}^{x+(1-\frac{\alpha}{2})t} (\frac{\alpha}{2}+1)^{-k} C(\xi,\sigma)d\sigma + \ldots \quad ,$$

$$\eta = \eta(x,t) = x+(1-\frac{\alpha}{2})t + \frac{\varepsilon^2}{16}\int_{x-(1+\frac{\alpha}{2})t}^{x+(1-\frac{\alpha}{2})t} (1-\frac{\alpha}{2})^{-k} C(\sigma,\eta)d\sigma + \ldots \quad . \tag{5.49}$$

Finally, we have to solve, say by successive approximations,

$$\xi = \xi(x,t) = x-(1+\frac{\alpha}{2})t + \frac{\varepsilon^2 t}{8} F(\xi) + 0(\varepsilon^2) \quad ,$$

$$\eta = \eta(x,t) = x+(1-\frac{\alpha}{2})t + \frac{\varepsilon^2 t}{8} E(\eta) + 0(\varepsilon^2) \quad , \tag{5.50}$$

where

$$E_{(\eta)} = (1-\frac{\alpha}{2})^{-k}[(1+\frac{\alpha}{2})^2 m(\eta)^2 + (1-\frac{\alpha}{2})^2 \frac{\Pi^2}{2}] \quad ,$$

$$F_{(\xi)} = (1+\frac{\alpha}{2})^{-k}[(1-\frac{\alpha}{2})^2 m(\xi)^2 + (1+\frac{\alpha}{2})^2 \frac{\Pi^2}{2}] \quad . \tag{5.51}$$

It is to be expected that, in the neighbourhood of t_b , there will be difficulty in solving expressions such as (5.50) because η_x η_t are unbounded there.

Thus we have $\xi(x,t)$, $\eta(x,t)$, and combining this with the results $u(\xi,\eta)$, $v(\xi,\eta)$, (5.32), (5.33) we have $u(x,t)$, $v(x,t)$, which one may integrate to obtain $Y(x,t)$.

These results are based on perturbations on the linear problem. If t is small and ε is small, clearly the approximation is good. However, if t is of the same order of magnitude as $\frac{1}{\varepsilon^2}$, we must examine to what extent boundary conditions are satisfied.

In the linear case, the line $x(\xi,\eta) = 0$ has the equation

$$(1 + \frac{\alpha}{2})\eta + (1 - \frac{\alpha}{2})\xi = 0 \quad . \tag{5.52}$$

But (5.46) shows that these values are altered by a term which for large values of $(\eta-\xi)$ could have considerable effect. If we consider a point $P(\xi_1,\eta_1)$ lying at the intersection of (5.52) with $\eta - \xi = \frac{k}{\varepsilon^2}$, where k is chosen depending on the value of t_b which we consider later, we find the deviation of $x(\xi_1,\eta_1)$ from $x = 0$. With the condition (5.52), and $m(x)$ taken from equation (5.10), and the fact that $f(x) = \varepsilon m(x)$, we integrate (5.46) to obtain:

$$x(\xi_1,\eta_1) = \frac{\varepsilon^2}{32}(\eta_1-\xi_1)\{(\frac{\alpha}{2} - 1)[(1 - \frac{\alpha}{2})^2 m(\xi_1)^2 + \frac{\Pi^2}{2}(1 + \frac{\alpha}{2})^2]$$

$$+ (\frac{\alpha}{2} + 1)[(1 + \frac{\alpha}{2})^2 m(\eta_1)^2 + \frac{\Pi^2}{2}(1 - \frac{\alpha}{2})^2]\} + 0(\varepsilon^2)$$

$$k = 0 \quad . \tag{5.53}$$

We note that $f(\eta_1) = f(\xi_1)$ along (5.52) and take

$$\eta_1-\xi_1 = \frac{16}{\varepsilon^2\Pi^3}\frac{(1 - \frac{\alpha}{2})}{(1 + \frac{\alpha}{2})^3} \quad , \tag{5.54}$$

so that

$$|x(\xi_1,\eta_1)| \leq \frac{(2-\alpha)(\alpha^3+10\alpha)}{2\Pi(2+\alpha)^3} < .08 \quad , \qquad 0 < \alpha < 2 \quad . \tag{5.55}$$

Similarly, for $k = -1$, we set $(\eta_1-\xi_1) = \dfrac{16}{\varepsilon^2\Pi^3\,(1 + \frac{\alpha}{2})^3}$ and we

calculate $x(\xi_1, \eta_1) < .05$ for $0 < \alpha < 2$.

Thus it seems we have a reasonably close approximation to the given boundary line. The analysis for $x = 1$ is similar. We note further that along (5.52) we have $f(\xi) = f(\eta)$, so that from the expressions for $v(\xi,\eta)$ in (5.33) and (5.34), we have $v = 0$ along that line.

Thus if we accept the first approximation to $x = 0$, we see that our boundary conditions are satisfied exactly. If, on the other hand, we wish to proceed to the second approximation to $x = 0$, this will not be so. In this case, by the continuity of v , it is clear that v will not differ appreciably from zero. However the value of Y may. This point would appear to require further study.

Now, we have breakdown if

$$J = [P-Q]t_\xi t_\eta = 0 \quad . \tag{5.56}$$

From (5.44) we have

$$t_\eta(\xi,\eta) = \frac{1}{2} + \frac{\varepsilon^2}{32}[(1+\frac{\alpha}{2})^{-k}C(\xi,\eta)+(1-\frac{\alpha}{2})^{-k}C(\eta,\eta)]$$
$$+ \frac{\varepsilon^2}{32}\int_\xi^\eta (1-\frac{\alpha}{2})^{-k}\, C_\eta(\sigma,\eta)d\sigma + 0(\varepsilon^2) \tag{5.57}$$

where

$$C_\eta(\sigma,\eta) = 2(1+\frac{\alpha}{2})[(1+\frac{\alpha}{2})m(\eta) + (1+\frac{\alpha}{2})m(\sigma)]m'(\eta) \quad . \tag{5.58}$$

Thus

$$t_\eta = -\frac{1}{2} + \frac{\varepsilon^2}{32}(1-\frac{\alpha}{2})^{-k}(1+\frac{\alpha}{2})^2 m(\eta)m'(\eta)(\eta-\xi) + 0(\varepsilon^2) \quad , \tag{5.59}$$

and we have $t_\eta = 0 + O(\varepsilon^2)$ when

$$\eta-\xi = \frac{16}{\varepsilon^2}\left[(1-\frac{\alpha}{2})^{-k}(1+\frac{\alpha}{2})^2 m(\eta)m'(\eta)\right]^{-1} \quad . \tag{5.60}$$

The factor $m'(\eta)$ takes on three forms which are given by (5.18). In order to obtain the maximum value for $m(\eta)m'(\eta)$, we consider the form

$$m(\eta) = \Pi \cos\left(\Pi \frac{j(1+\gamma)-\eta}{\gamma}\right) \tag{5.61}$$

$$j(1+\gamma)-\gamma < \eta < j(1+\gamma) \qquad j = 1,2,\ldots$$

so that

$$m(\eta)m'(\eta) = \frac{\Pi^3}{2\gamma}\sin\left(2\Pi \frac{j(1+\gamma)-\eta}{\gamma}\right)$$

$$\leq \frac{\Pi^3}{2\gamma} \quad . \tag{5.62}$$

We employ the approximation $t = t^o = \frac{\eta-\xi}{2}$ to determine the time of breakdown t_b, and we take the minimum value of $\eta-\xi$ satisfying (5.60); thus

$$t_b = \frac{8(1-\frac{\alpha}{2})^{1+k}}{\varepsilon^2\Pi^3(1+\frac{\alpha}{2})^3} \quad , \qquad k = 0,-1 \quad . \tag{5.63}$$

A consideration of the term $t^{(1)}$ in (5.44) shows that

$$|\varepsilon^2\, t^{(1)}| < 2 \quad , \tag{5.64}$$

for time t_b, so that where ε is small, $t_b = O(\frac{1}{\varepsilon^2})$, and (5.63) is significantly representative when breakdown occurs.

Finally, we indicate a procedure for determining the solution $Y(x,t)$ to the problem. Consider a point $P(x,t)$ where $0 < t < t_b$ and

$$\eta(P_o) = -\gamma\xi(P) \quad . \tag{5.68}$$

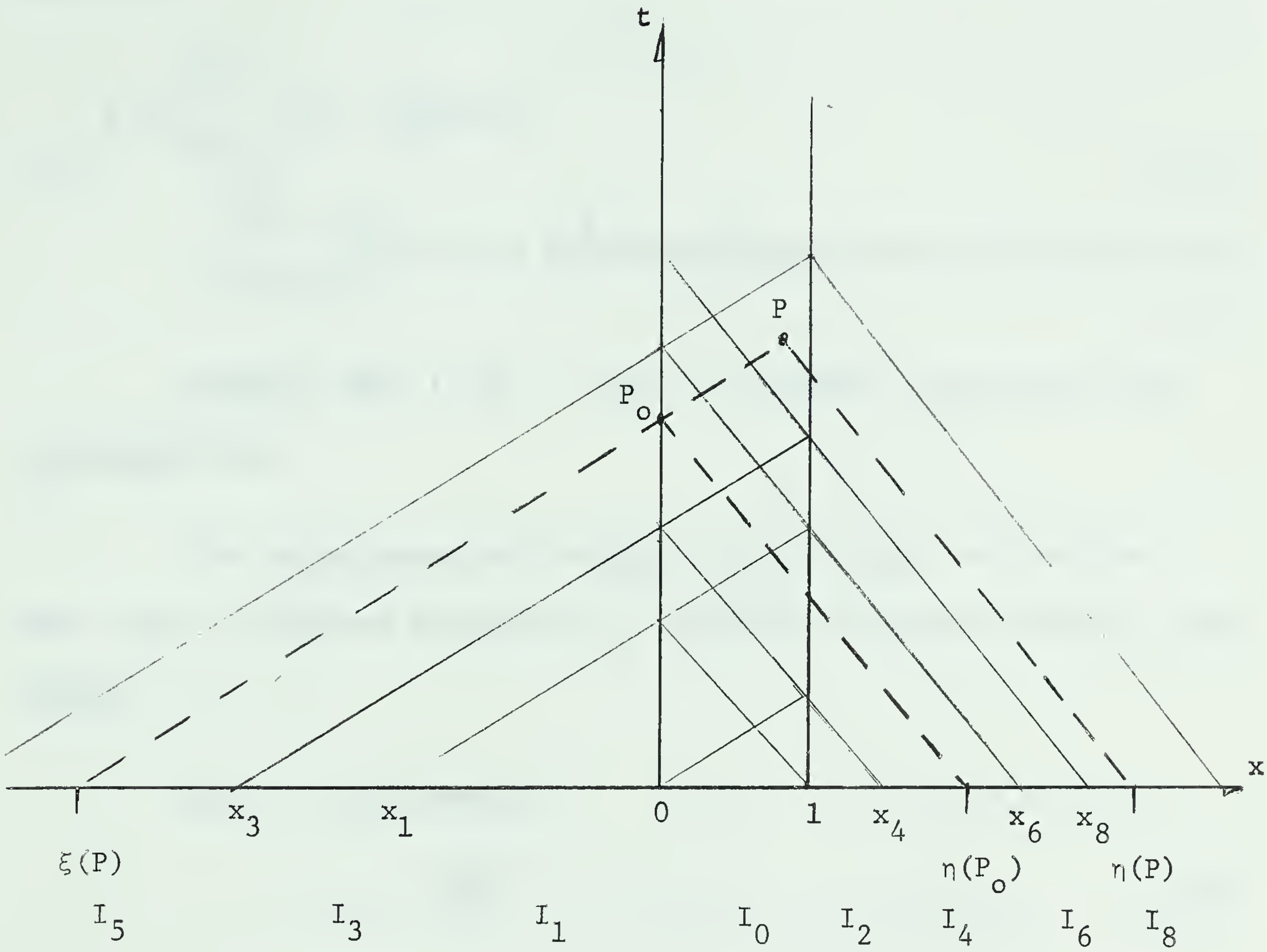


Figure XIII

Now, the range of integration of (5.65), from $\eta(P_o)$ to $\eta(P)$ is not large, so it suits us to consider only the first terms of each series expression. We perform the sample solution for the first threadline problem, $k = 0$;

$$u = \frac{\varepsilon}{2}[(1 + \frac{\alpha}{2})m(\eta) + (1 - \frac{\alpha}{2})m(\xi)]$$

$$v = \frac{\varepsilon}{2}(1 - \frac{\alpha^2}{4})[m(\eta)-m(\eta)]$$

$$x_\eta = \frac{1}{2}(1 + \frac{\alpha}{2}) + \frac{\varepsilon^2}{16}(\eta-\xi)(\frac{\alpha}{2} + 1)^3 m(\eta)m'(\eta)$$

$$t_\eta = \frac{1}{2} + \frac{\varepsilon^2}{16}(\eta-\xi)(\frac{\alpha}{2} + 1)^2 m(\eta)m'(\eta) \quad . \tag{5.69}$$

We set $Y(P_o) = 0$ because of the desired boundary conditions and thus

$$Y = \int_{\eta(P_o)}^{\eta(P)} \frac{\varepsilon}{2}(1 + \frac{\alpha}{2})m(\delta)d\delta$$
$$+ \int_{\eta(P_o)}^{\eta(P)} \frac{\varepsilon^3}{32}(\delta-\xi)(1 + \frac{\alpha}{2})^4[(1+\gamma)m^2(\delta)m'(\delta)+2\gamma m(\xi(P))m(\delta)m'(\delta)]d\delta \quad . \tag{5.70}$$

Clearly, for $t < \frac{1}{\varepsilon}$, when ε is small, the second term is insignificant.

The calculations are straight forward except for the fact that $m(x)$ is defined differently on different intervals; that is, from (5.18)

$$\begin{aligned} m(x) &= \Pi \cos \Pi(x-x_i) \quad , \quad x \in I_i \ , \quad i = 3,4,7,8,11,12,\ldots \\ &= \Pi \cos \Pi \frac{x_i-x}{\gamma} \quad , \quad x \in I_i \ , \quad i = 2,6,10,\ldots \\ &= \Pi \cos \Pi\gamma(x-x_i) \quad , \quad x \in I_i \ , \quad i = 1,5,9,\ldots \quad . \end{aligned} \tag{5.71}$$

Let us suppose that, as in Figure XIII, $\eta(P) \in I_8$, $\eta(P_o) \in I_4$, so that we must consider the integral for each interval I_4, I_6, I_8 . From (5.67) and (5.68),

$$\begin{aligned} \eta(P) &= x + (1 - \frac{\alpha}{2})t \\ \eta(P_o) &= -\gamma(x - (1 + \frac{\alpha}{2})t \quad , \end{aligned} \tag{5.72}$$

and from (5.70),

$$Y = \int_{\eta(P_o)}^{x_6} \Pi \cos \Pi(x-x_4)dx + \int_{x_6}^{x_8} \Pi \cos \Pi \frac{x_6-x}{\gamma}$$

$$+ \int_{x_8}^{\eta(P)} \Pi \cos \Pi(x-x_8) \quad , \tag{5.73}$$

where

$$x_i = \frac{i}{4}(1+\gamma) \quad , \qquad i = 4,8,\ldots \quad ,$$
$$x_i = 1 + \frac{i-2}{4}(1+\gamma) \quad , \qquad i = 2,6,10,\ldots \quad . \tag{5.74}$$

As

$$\sin \Pi\, x_i = 0 \quad , \qquad i = 1,2,3,\ldots$$

there remains only

$$Y = \sin \Pi(x+(1-\frac{\alpha}{2})t-x_4)$$

$$- \sin \Pi(-\gamma(x-(1+\frac{\alpha}{2})t)-x_8) \quad . \tag{5.75}$$

In general then, computations involve a consideration of the particular intervals over which calculations are made.

CHAPTER VI

NUMERICAL RESULTS

The object of the present chapter is to calculate and plot various members of the families of characteristic lines. The initial intersection of two members of the same family of lines indicates that $t_\xi = 0$ or $t_\eta = 0$ at the point, thus we have $J = 0$ and the occurrence of breakdown. For the fixed string problem, we have plotted a few examples to indicate the path of the lines and the shape of the singular curves where $J = 0$, and we have compared the breakdown times obtained in this way, to the analytical results of Chapter IV.

Of the methods available, the method of Masseau [8] was selected for its simplicity, and for its accuracy in view of the slow variation in the slope of the characteristic lines. The method consists in a calculation of points where characteristic lines intersect, and thus provide an approximation to the path of characteristic lines by joining the points.

Consider a system of two first order equations in two dependent and two independent variables (2.1). Having written it in terms of characteristic parameters (2.5), (2.6) we replace the derivatives with expressions such as:

$$u_\xi = \frac{u(P_1) - u(P_2)}{|P_1 - P_2|} , \qquad (6.1)$$

and thus obtain

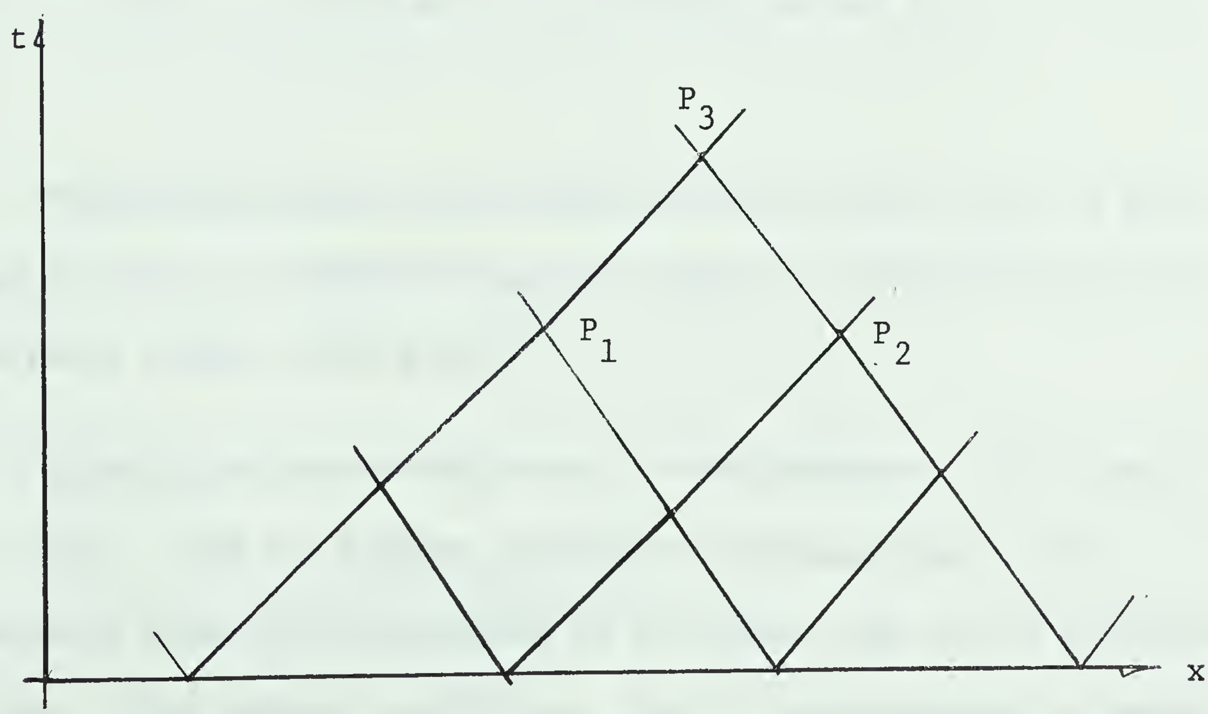
$$a\frac{u(P_3) - u(P_2)}{|P_3 - P_2|} + b\frac{v(P_3) - v(P_2)}{|P_3 - P_2|} = 0$$

$$c\frac{u(P_3) - u(P_1)}{|P_3 - P_1|} + d\frac{v(P_3) - v(P_1)}{|P_3 - P_1|} = 0 \tag{6.2}$$

$$\frac{x(P_3) - x(P_2)}{|P_3 - P_2|} + e\frac{t(P_3) - t(P_2)}{|P_3 - P_2|} = 0$$

$$\frac{x(P_3) - x(P_1)}{|P_3 - P_1|} + f\frac{t(P_3) - t(P_1)}{|P_3 - P_1|} = 0 \quad , \tag{6.3}$$

where a, b, c, d, e, and f are functions of u, v, x, and t , P_1, P_2, and P_3 are ajacent grid points as in Figure XIV and $|P_i - P_j|$ is the distance between the two points.



LINE SEGMENT ARRAY WHICH APPROXIMATES CHARACTERISTIC CURVES

Figure XIV

Consider two points P_1, P_2, (such as points of the initial line), where u, v, x, t are known. Calculating a, b and e at P_2, and c, d and f at P_1, we solve (6.2) and (6.3) for $u(P_3)$, $v(P_3)$, $x(P_3)$, and $t(P_3)$. This serves as a first approximation to the values for the point of intersection of the lines through P_1 and P_2

Then we take

$$e = \frac{1}{2}\,[e(P_3) + e(P_2)]$$
$$f = \frac{1}{2}\,[f(P_3) + f(P_1)] \tag{6.4}$$

and by (6.3) calculate new values for $x(P_3)$, $t(P_3)$. Using similar averaging for a, b, c, and d which are calculated with the most recent values of u, v, x, and t, we solve for $u(P_3)$, $v(P_3)$ from (6.2). Alternating these last two steps, the solution of $x(P_3)$, $t(P_3)$, and of $u(P_3)$, $v(P_3)$, should give a convergent series for u, v, x, t at P_3.

Proceeding from a set of points on the initial line, a grid of points is built up approximating the points of intersection of the characteristic lines. (Figure XIV).

In the problems we deal with, we can determine $u = u(r+s) = u(r(\xi) + s(\eta))$ from the Riemann invariants corresponding to the characteristic lines which intersect at the point, and we can determine $r(\xi)$, $s(\eta)$ from initial conditions. So it is unnecessary to employ the above iterative process in order to determine u, and thus the

slopes $-e^{-1}$, $-f^{-1}$ of the characteristic lines, which are functions of u only, are known. If we take the average slope (6.4), of the two end points of the line segment, we can calculate $x(P_3)$, $t(P_3)$ immediately.

The inversion of

$$r + s = \int_o^u (1 + \varepsilon u^p)^{q/2} \quad , \tag{6.5}$$

to provide $u = u(r+s)$ cannot be carried out exactly, except for $p = 1$. But for the finite number of discrete values r and s which enter into the calculations, we can use Einstein's method to approximate a value of u for which

$$f(u) = r + s \tag{6.6}$$

where

$$f(u) = u + \frac{q}{2}\,\varepsilon\,\frac{u^{p+1}}{p+1} + \frac{q}{2}\left(\frac{q}{2} - 1\right)\frac{\varepsilon^2 u^{2p+1}}{2!} + \ldots \quad . \tag{6.7}$$

The Einstein approximation is simple. As

$$(r+s)_u = (1 + \varepsilon u^p)^{q/2} \quad , \qquad |u| < \frac{1}{\varepsilon^p} \quad , \tag{6.8}$$

the function $r+s$ is monotonic in u. Incrementing u by

$$\Delta u = \frac{r+s - f(u)}{(1 + \varepsilon u^p)^{q/2}} \quad , \tag{6.9}$$

gives a convergent sequence of values of u. The degree of accuracy

of the numerical results is increased by taking a denser set of characteristic lines, and thus shortening the straight line segements. As the slopes of the lines vary slowly, the nonlinearity being a factor of ε, the straight line segments with mean slope are a good approximation to the curve. An indication of the accuracy of the numerical results was found in a comparison of the results obtained when the number of initial points in $0 \leq x \leq 2$ were given the two values 64 and 80. The resulting times of breakdown differed by at most .03, and corresponding x coordiantes differed by at most .003. Thus the numerical results would appear to be satisfactory.

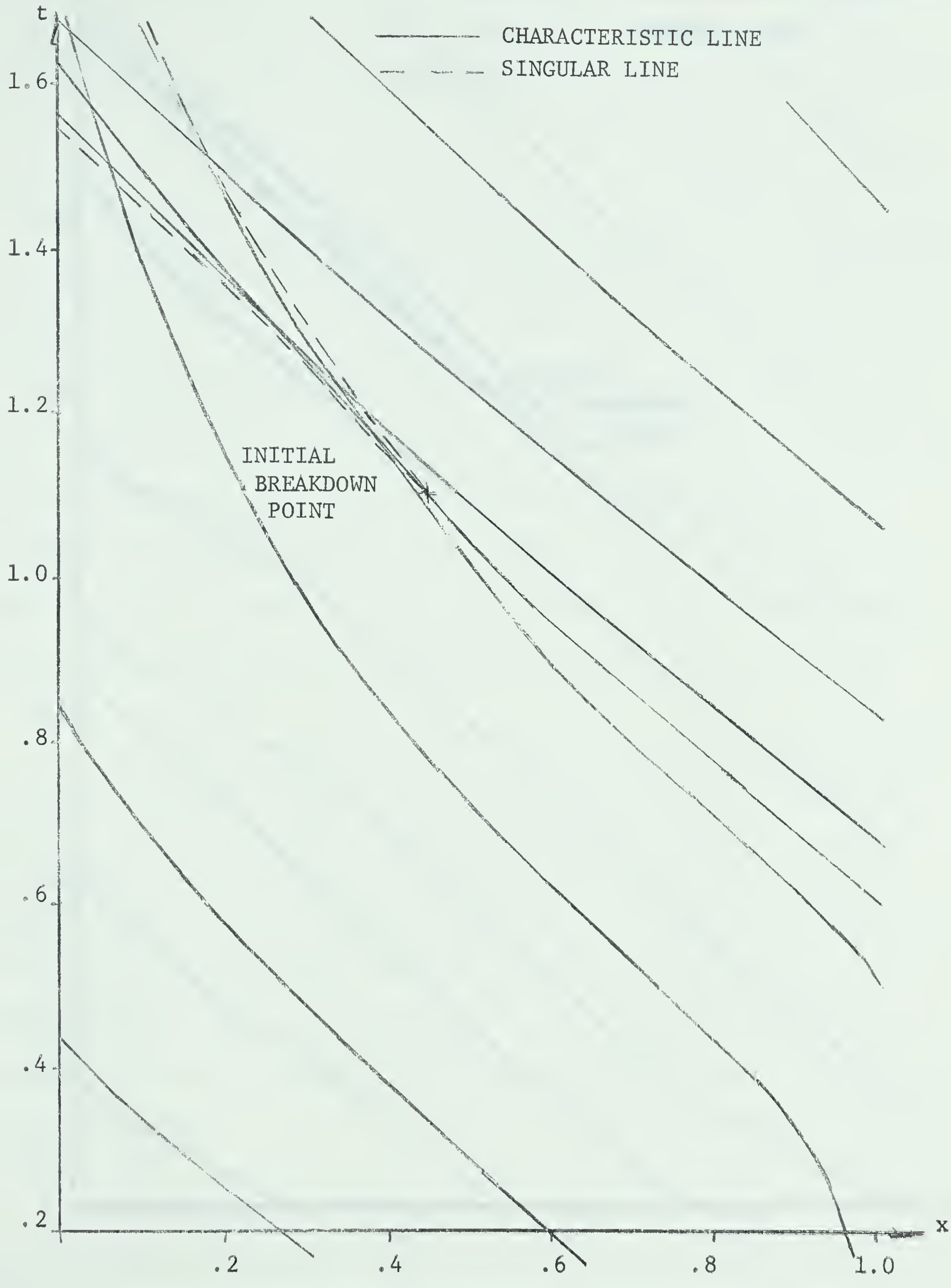
The following is a comparison of the values of ξ, η, t at initial breakdown, as arrived at by the analysis of Chapter IV, and the numerical work on the fixed string problem.

For $q = 1$ and $n = 1$

$p = 1$		Analytical			Numerical		
ε	a	ξ	η	t	ξ	η	t
.3	1	.5	3.2	1.45	.54	2.58	1.09
.1	1	.5	8.62	4.06	.50	8.50	3.97
.01	2	.5	41.10	20.50	.50	42.50	20.90
$p = 2$							
.3	1	.25	1.97	.86	.15	2.62	.91
.1	1	.25	5.41	2.38	.18	5.26	2.67
.01	2	.25	13.15	6.45	.25	16.10	7.64

TABLE

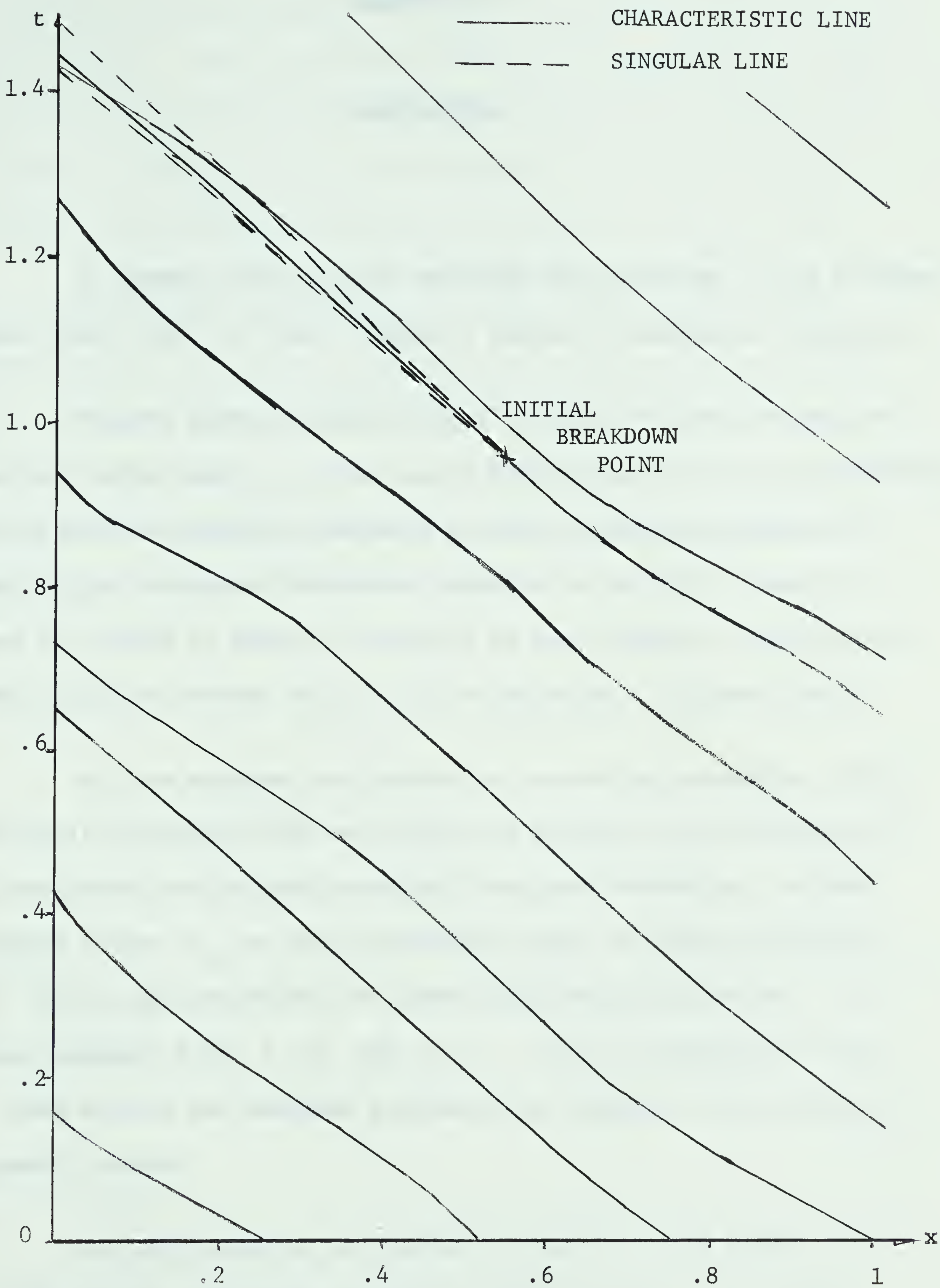
As is seen from the tabulated results, for fairly large ε , when for $p = 1$, the results for numerical and analytical calculation of breakdown are close, except where ε is very close to $\frac{1}{(an\Pi)^p}$. For $p = 2$ the linear approximation to t_b is representative, but less accurate. The two graphs (Figures XV and XVI) indicate the curves that the characteristic lines follow. The intersection of the lines indicates breakdown.



COMPUTER PLOT OF CHARACTERISTIC LINES FOR THE FIXED STRING

$\varepsilon = .3, \quad a = 1, \quad p = 1, \quad q = 1, \quad n = 1$

Figure XV



COMPUTER PLOT OF CHARACTERISTIC LINES FOR FIXED STRING

$\varepsilon = .3, \quad a = 1, \quad p = 2, \quad q = 1, \quad n = 1$

Figure XVI

CHAPTER VII

CONCLUSIONS

It appears that we have indicated the solutions to the problems we have dealt with, but there remains a number of unanswered questions.

We have found a solution to (1.12) for the fixed string, and shown that after time t_b , the second derivatives of Y , are unbounded. Thus the solution cannot be extended smoothly beyond such points. As a result, the recurrence phenomenon observed in the F.P.U. report [7] cannot be studied by means of the model we have studied. Further work in this has been carried out in [16] by employing a different model.

We have examined two models for the moving threadline, using the initial conditions that would give the required end conditions for the unperturbed problem with straight line characteristics. We have estimated a time t_b at which breakdown occurs in these problems as well. But as we have shown, the lines along which we have set $v = 0$ deviate somewhat from $x = 0$ and $x = 1$. This is indicative of the fact that we have not examined precisely the boundary value problem we originally stated.

An improvement on our approach would be, as we stated, to vary the initial conditions slightly so that we have $v = 0$ along precisely $x = 0$ and $x = 1$. This would involve beginning with the interval $[0,1]$, and determining the pairs of characteristic lines

which intersect at the boundaries to a greater degree of accuracy than we have already. Then the initial condition is assigned so that the Riemann invariants combine as required to give $v = 0$. Difficulties in actually carrying out these calculations have prevented us from employing this procedure and the matter is open to further study.

The numerical calculations in the F.P.U. report [7] and Zaiser's Thesis [18] failed to indicate the singularities we predict in our analytical and numerical results. This is attributed to the nature of their approaches which tend to smooth out discontinuities. Rather, it is an approach such as ours, which examines the behaviour of characteristic lines, which is successful.

APPENDIX I

EQUIVALENCE OF THE TWO FORMS OF PERTURBATION

Consider the two methods of perturbation we have used. Firstly we consider problems of the form

$$\begin{aligned} &u_{tt} = F(\varepsilon u_x, \varepsilon u_t) u_{xx} \\ &u(x,0) = f(x) \\ &u_t(x,0) = 0 \quad , \end{aligned} \tag{A.1.1}$$

and secondly of the form

$$\begin{aligned} &v_{tt} = F(v_x, v_t) v_{xx} \\ &v(x,0) = \varepsilon f(x) \\ &v_t(x,0) = 0 \quad . \end{aligned} \tag{A.1.2}$$

We assume that for all values of x, t, ε considered that solutions do exist, and are unique. Then these two problems are equivalent in the sense that given a solution of one of them then the solution of the other may be found by the relation

$$v = \varepsilon u \quad , \tag{A.1.3}$$

unless of course $\varepsilon = 0$. In this case it is obvious that $v \equiv 0$ and if $\lim_{\varepsilon \to o} u(\varepsilon,x,t) = u(0,x,t)$ then

$$o = \lim_{\varepsilon \to o} v(\varepsilon,x,t) = \lim_{\varepsilon \to o} \varepsilon\, u(\varepsilon,x,t) \quad .$$

It is not clear however that

$$\lim_{\varepsilon \to o} u(\varepsilon,x,t) = \lim_{\varepsilon \to o} \frac{1}{\varepsilon} v(\varepsilon,x,t) \quad ,$$

and in this sense the equivalence may fail.

APPENDIX II

CONSIDERATION OF THE SUCCESSIVE APPROXIMATION RESULTING FROM (2.37)

Consider equation (2.37) and assume that we have

$$\overline{P} = \sum_{k=1}^{\infty} \varepsilon^{k-1} P^{(k)} \quad ,$$
$$\overline{Q} = \sum_{k=1}^{\infty} \varepsilon^{k-1} Q^{(k)} \quad . \tag{A.2.1}$$

If we substitute these series in the integral equation (2.38) we find that

$$t = \frac{\eta-\xi}{P^o-Q^o} - \frac{1}{P^o-Q^o} \int_{\xi}^{\eta} \sum_{k=1}^{\infty} \varepsilon^k \, [P^{(k)}(\sigma,\eta)t_{\sigma}(\sigma,\eta)+Q^{(k)}(\xi,\sigma)t_{\sigma}(\xi,\sigma)]d\sigma \quad . \tag{A.2.2}$$

The expression for t in [16] is a power series in ε. If we set $t = \sum_{n=o}^{\infty} \varepsilon^n t^{(n)}$ in (A.2.2), and formally rearrange the terms, we obtain

$$t = \frac{\eta-\xi}{P^o-Q^o} - \frac{1}{P^o-Q^o} \int_{\xi}^{\eta} \sum_{n=1}^{\infty} \varepsilon^n \sum_{k=1}^{n} [P^{(k)}(\sigma,\eta)t_{\sigma}^{(n-k)}(\sigma,\eta)+Q^{(k)}(\xi,\sigma)t_{\sigma}^{(n-k)}(\xi,\sigma)]d\sigma \quad . \tag{A.2.3}$$

Provided now that the series is uniformly convergent for values of ξ,η considered we can interchange the order of summation and integration to yield (2.32). The convergence of series concerned is considered in Appendix III.

APPENDIX III

CONVERGENCE OF THE SERIES FOR t, x, t_ξ, t_η

We will show that the series for t, in (2.31), converges under the conditions set out below. Our aim is not to find the most general conditions. For simplicity in the following we assume $\eta-\xi \geq 0$.

First we note that

$$\int_\xi^\eta P^{(k)}(\xi,\sigma)t_\sigma^{(n)}(\xi,\sigma)d\sigma = P^{(k)}(\xi,\eta)t^{(n)}(\xi,\eta) - \int_\xi^\eta P_\sigma^{(k)}(\xi,\sigma)t^{(n)}(\xi,\sigma)d\sigma \tag{A.3.1}$$

We begin with the expression given by

$$t^{(k)} = [P^o-Q^o]^{-1}\left\{\int_\xi^\eta \sum_{n=1}^{k} [P_\sigma^{(n)}(\xi,\sigma)t^{(k-n)}(\xi,\sigma)+Q_\sigma^{(n)}(\sigma,\eta)t^{(k-n)}(\sigma,\eta)]d\sigma \right.$$
$$\left. - \sum_{n=1}^{k} [P^{(n)}(\xi,\eta)-Q^{(n)}(\xi,\eta)]t^{(k-n)}(\xi,\eta)\right\} . \tag{A.3.2}$$

We define $S_N = \sum_{k=o}^{N} \varepsilon^k \, |t^{(k)}|$, and note that for given quantities $a^{(k)}$, that

$$\sum_{n=1}^{N} \sum_{k=1}^{n} \varepsilon^n a^{(k)}|t^{(n-k)}| = \sum_{k=1}^{N} \sum_{n=k}^{N} \varepsilon^n a^{(k)}|t^{(n-k)}|$$
$$= \sum_{k=1}^{N} \varepsilon^k a^{(k)} S_{N-k} . \tag{A.3.3}$$

Now consider the sum S_N . We have

$$S_N \leq \left|\frac{\eta-\xi}{P^o-Q^o}\right| + \frac{1}{|P^o-Q^o|}\left\{\sum_{k=1}^{N}\sum_{n=1}^{k} \varepsilon^k |P^{(n)}(\xi,\eta)-Q^{(n)}(\xi,\eta)||t^{(k-n)}(\xi,\eta)|\right.$$

$$\left. + \sum_{k=1}^{N} \varepsilon^k \int_\xi^\eta \sum_{n=1}^{k} P_\sigma^{(n)}(\xi,\sigma)t^{(k-n)}(\xi,\sigma)+Q_\sigma^{(n)}(\sigma,\eta)t^{(k-n)}(\sigma,\eta)\ d\sigma\right\}$$

$$\leq \left|\frac{\eta-\xi}{|P^o-Q^o|}\right| + \frac{1}{|P^o-Q^o|}\left\{\sum_{n=1}^{N} \varepsilon^n \sum_{k=n}^{N} \varepsilon^{k-n}|P^{(n)}(\xi,\eta)-Q^{(n)}(\xi,\eta)||t^{(k-n)}|\right.$$

$$+ \sum_{n=1}^{N} \varepsilon^n \sum_{k=n}^{N} \varepsilon^{k-n} \int_\xi^\eta |P_\sigma^{(n)}(\xi,\sigma)t^{(k-n)}(\xi,\sigma)+Q_\sigma^{(n)}(\sigma,\eta)t^{(k-n)}(\sigma,\eta)|d\sigma$$

$$\left.\leq S_o + \frac{1}{|P^o-Q^o|} \sum_{n=1}^{N} \varepsilon^n|P^{(n)}(\xi,\eta)-Q^{(n)}(\xi,\eta)|S_{N-n}\right\}$$

$$+ \frac{1}{|P^o-Q^o|} \sum_{n=1}^{N} \int_\xi^\eta \varepsilon^n|P_\sigma^{(n)}(\xi,\sigma)|S_{N-n}+|Q_\sigma^{(n)}(\sigma,\eta)|S_{N-n}\ d\sigma . \qquad (A.3.4)$$

S_N is monatonically increasing of N so that

$$S_{N-n} \leq S_{N-1}\ , \qquad n \geq 1 \quad .$$

We further stipulate that we are considering those values of ε for which the series for P and Q are absolutely convergent, and assume a range of values of ξ,η such that

$$\sum_{k=1}^{\infty} \varepsilon^k\ |P^{(k)}| \leq A\ ,$$

$$\sum_{k=1}^{\infty} \varepsilon^k\ |Q^{(k)}| \leq B\ ,$$

$$\sum_{k=1}^{\infty} \varepsilon^k\ |P_\xi^{(k)}| \leq A'\ , \qquad (A.3.5)$$

$$\sum_{k=1}^{\infty} \varepsilon^k\ |Q_\xi^{(k)}| \leq B' \quad .$$

Thus

$$S_N \leq S_o + \frac{1}{|P^o - Q^o|} \left\{ (A+B) S_{N-1} + \int_{\xi}^{\eta} (A' S_{N-1} + B' S_{N-1}) d\sigma \right\} . \tag{A.3.6}$$

It remains to show that there exists a K such that

$$S_N \leq K\, e^{\eta-\xi} \quad .$$

We do this inductively. First we note that:

$$S_o = \frac{1}{|P^o - Q^o|} < \frac{e^{\eta-\xi}}{|P^o - Q^o|} \quad . \tag{A.3.7}$$

Then we assume that $K \geq |P^o - Q^o|^{-1} > 0$, and further that

$$S_n \leq K\, e^{\eta-\xi} \quad , \qquad n \leq N-1 \quad .$$

From (A.3.6) we have

$$\begin{aligned} S_N \leq\ & |P^o - Q^o|^{-1} \{ (\eta-\xi) + (A+B) K\, e^{\eta-\xi} \\ & + \int_{\xi}^{\eta} A'\, K\, e^{\sigma-\xi} + B'\, K\, e^{\eta-\xi} d\sigma \} \quad , \end{aligned} \tag{A.3.8}$$

and clearly

$$S_N \leq |P^o - Q^o|^{-1} \{ 1 + K(A+B+A'+B')\, e^{\eta-\xi} \} \quad . \tag{A.3.9}$$

We require a $K > 0$ such that

$$\begin{aligned} & 1 + K(A+B+A'+B') < K\, |P^o - Q^o| \\ & K \geq [\, |P^o - Q^o| - (A+B+A'+B')]^{-1} \quad . \end{aligned} \tag{A.3.10}$$

Thus, the required K exists under the condition that

$$(A+b+A'+B') < |P^o - Q^o| \quad . \tag{A.3.11}$$

Clearly, we can choose ε small enough so that this holds.

Thus, provided $\eta-\xi$ is bounded, the series is bounded. As S_N is monotonic and bounded, it must converge to a finite limit, for each value of (ξ,η) . We can show in a similar way that the series for x, t_ξ and t_η converge under the same conditions.

APPENDIX IV

ORDER OF OMITTED TERMS IN SETTING $t_\xi = t_\xi^0 + t_\xi^{(1)}$.

Our assumption in predicting the time of breakdown is that

$$\sum_{k=2}^{\infty} \varepsilon^k \, t_\xi^{(k)} < \varepsilon K \quad , \tag{A.4.1}$$

for some constant K . If a specific representation of $t(\xi,\eta)$ is known, as in Zabusky [15], it is possible to check the assumption (3.7). We shall attempt to verify our procedure here with respect to the problem

$$\begin{aligned} y_{tt} &= (1 + \varepsilon y_x) y_{xx} \\ y(x,0) &= a \sin \Pi x \quad . \\ y(0,t) &= y(1,t) = 0 \end{aligned} \tag{A.4.2}$$

If we reduce this problem to a set of equations as in Chapter II, it follows that, if we set

$$f(x) = (1 + \varepsilon a \Pi \cos \Pi x)^{\frac{1}{2}} \quad , \tag{A.4.3}$$

then we find the equation

$$\begin{aligned} & t_{\eta\xi} + \frac{f^2(\eta)\, f'(\eta)}{f^3(\xi) + f^3(\eta)} \, t_\xi + \frac{f^2(\xi)\, f'(\xi)}{f^3(\xi) + f^3(\eta)} \, t_\eta = 0 \\ & t(\xi,\xi) = 0 \\ & t_\xi(\xi,\xi) = - t_\eta(\xi,\xi) = - \frac{1}{2} f(\xi) \quad . \end{aligned} \tag{A.4.4}$$

Again if we write $t = T + \Theta$ with

$$T(\xi,\eta) = \frac{1}{2}\int_{\xi}^{\eta} \frac{d\mu}{f(\mu)} , \tag{A.4.5}$$

then

$$\begin{aligned} &T_{\xi\eta} = 0 \\ &T(\xi,\xi) = 0 \\ &T_{\xi}(\xi,\xi) = -T_{\eta}(\xi,\xi) = -\frac{1}{2} f(\xi) \end{aligned} \tag{A.4.6}$$

and

$$\begin{aligned} &\Theta_{\xi\eta} + \frac{f^2(\xi)f'(\eta)}{f^3(\xi)+f^3(\eta)}(\Theta_{\xi}+T_{\xi}) + \frac{f^2(\xi)f'(\xi)}{f^3(\xi)+f^3(\eta)}(\Theta_{\eta}+T_{\eta}) = 0 , \\ &\Theta(\xi,\xi) = \Theta_{\xi}(\xi,\xi) = \Theta_{\eta}(\xi,\xi) = 0 . \end{aligned} \tag{A.4.7}$$

Rather than proceed as before, we shall now use successive approximations as in Garabedian's book [10]. We set

$$\Theta_o = 0$$

$$\Theta_1 = \frac{1}{2} \iint_R \left\{ \frac{1}{f(\xi)} \frac{f^2(\eta)f'(\eta)}{f^3(\xi)+f^3(\eta)} - \frac{1}{f(\eta)} \frac{f^2(\xi)f'(\xi)}{f^3(\xi)+f^3(\eta)} \right\} d\xi d\eta \tag{A.4.8}$$

$$\Theta_{n+1} = \Theta_1 - \iint_R \left\{ \frac{f^2(\eta)f'(\eta)}{f^3(\xi)+f^3(\eta)} p_n + \frac{f^2(\xi)f'(\xi)}{f^3(\xi)+f^3(\eta)} q_n \right\} d\xi d\eta , \quad n > 1 \tag{A.4.9}$$

where $p_n = \frac{\partial\Theta_n}{\partial\xi}$, $q_n = \frac{\partial\Theta_n}{\partial\eta}$ and R is the region in the interior of triangle ABP in Figure IV . Then

$$t_n = T + \Theta_n$$

$$= T + \Theta_1 + (\Theta_n - \Theta_1)$$

and

$$\frac{\partial t_n}{\partial \xi} = T_\xi + p_1 + (p_n - p_1) \quad . \qquad \text{(A.4.10)}$$

It is well known that for a linear problem such as this one that the approximations do converge to a solution to the problem. We shall attempt to show that, if $\varepsilon|\eta-\xi|$ is bounded, that is, if $\eta-\xi = 0(\frac{1}{\varepsilon})$ then

$$|p_n - p_1| = 0(\varepsilon) \qquad n = 2,3,\ldots \quad . \qquad \text{(A.4.11)}$$

In this case our condition

$$\frac{\partial t}{\partial \xi} = 0$$

will, to the order we have been working to, be given those values of (ξ,η) such that

$$\frac{\partial T}{\partial \xi} + p_1 = 0 \quad . \qquad \text{(A.4.12)}$$

We obtain, after a short calculation,

$$p_1(\xi,\eta) = \frac{-1}{6f(\xi)} \ln\left\{\frac{f^3(\xi)+f^3(\eta)}{2f^3(\xi)}\right\} + \frac{f^2(\xi)f'(\xi)}{2}\int_\xi^\eta \frac{d\mu}{f(\mu)[f^3(\xi)+f^3(\mu)]} \quad ,$$

$$\text{(A.4.13)}$$

$$q_1(\xi,\eta) = \frac{1}{6f(\eta)} \ln\left\{\frac{f^3(\xi)+f^3(\eta)}{2f^3(\eta)}\right\} + \frac{f^2(\eta)f'(\eta)}{2}\int_\xi^\eta \frac{d\mu}{f(\mu)[f^3(\eta)+f^3(\mu)]} \quad .$$

We wish to find p_2-p_1 , q_2-q_1 , by substitution of these above quantities in (A.4.9). We shall break this into two parts P^I , Q^I

obtained from the logarithm terms in (A.4.13) and P^{II} , Q^{II} obtained from the integral terms. Again, after a little calculation we find

$$P^I = -\frac{1}{6}\int_\xi^\eta \left[\frac{f^2(\mu)f'(\mu)}{f(\xi)[f^3(\xi)+f^3(\mu)]} \ln\left\{\frac{f^3(\xi)+f^3(\mu)}{2f^3(\xi)}\right\} - \frac{f^2(\xi)f'(\xi)}{f(\mu)[f^3(\xi)+f^3(\mu)]} \ln\left\{\frac{f^3(\xi)+f^3(\mu)}{2f^3(\mu)}\right\}\right] d\mu \quad . \qquad \text{(A.4.14)}$$

$$Q^I = \frac{1}{6}\int_\xi^\eta \left[\frac{f^2(\eta)f'(\eta)}{f(\mu)[f^3(\eta)+f^3(\mu)]} \ln\left\{\frac{f^3(\eta)+f^3(\xi)}{2f^3(\mu)}\right\} - \frac{f^2(\mu)f'(\mu)}{f(\mu)[f^3(\eta)+f^3(\mu)]} \ln\left\{\frac{f^3(\eta)+f^3(\mu)}{2f^2(\mu)}\right\}\right] d\mu \quad . \qquad \text{(A4.15)}$$

We claim that there exists a positive constant K such that

$$|P^I| < \frac{\varepsilon^2 K}{2}|\xi-\eta| \quad ,$$
$$|Q^I| < \frac{\varepsilon^2 K}{2}|\xi-\eta| \quad . \qquad \text{(A.4.16)}$$

To see this we write, for example,

$$\ln\left\{\frac{f^3(\xi)+f^3(\mu)}{2f^3(\xi)}\right\} = \ln\left\{1+\frac{f^3(\mu)-f^3(\xi)}{2f^3(\xi)}\right\}$$

$$= \frac{f^3(\mu)-f^3(\xi)}{2f^3(\xi)} \cdot \frac{1}{1+\lambda} \quad , \quad 0 < \lambda < 1 \quad . \qquad \text{(A.4.17)}$$

Clearly $|f^3(\xi)|^{-1} > \frac{1}{2^{3/2}}$ if we ensure that $\varepsilon < \frac{1}{2a\Pi}$, as we do here in what follows. Further

$$|f^3(\mu)-f^3(\xi)| \leq \left|\frac{(1+\varepsilon a\Pi \cos \Pi\mu)^3 - (1+\varepsilon a\Pi \cos \Pi\xi)^3}{(1+\varepsilon a\Pi \cos \Pi\mu)^{3/2}+(1+\varepsilon a\Pi \cos \Pi\xi)^{3/2}}\right|$$

$$\leq \left|\frac{(1+\varepsilon a\Pi)^3 - (1-\varepsilon a\Pi)^3}{2(\frac{1}{2})^{3/2}}\right|$$

$$= \frac{6\varepsilon a\Pi + 2\varepsilon^3 a^3 \Pi^3}{2^{-(1/2)}} < K_1\varepsilon \quad , \tag{A.4.18}$$

for $K_1 > 0$, and ε sufficiently small. Noting also that

$$f'(x) = \frac{-\varepsilon a\Pi^2 \sin \Pi x}{(1+\varepsilon a\Pi \cos \Pi x)^{1/2}} \quad , \tag{A.4.19}$$

and that $(\frac{1}{2})^{1/2} < f(x) < 2^{1/2}$ for all x and all $\varepsilon < \frac{1}{2a\Pi}$ it is not difficult to see that the statement (A.4.16) is true for some $K > 0$.

We turn now to the contribution from the second part and find

$$P^{II} = \frac{1}{6} \int_\xi^\eta \frac{f^2(\xi)f'(\xi)}{f(\mu)[f^3(\xi)+f^3(\mu)]} \ln\left\{\frac{f^3(\xi)+f^3(\eta)}{f^3(\xi)+f^3(\mu)}\right\} d\mu$$

$$- \frac{1}{6} \int_\xi^\eta \frac{f^2(\xi)f'(\xi)}{f(\mu)[f^3(\mu)-f^3(\xi)]} \ln\left\{\frac{f^3(\mu)+f^3(\eta)}{f^3(\xi)+f^3(\eta)} \cdot \frac{f^3(\xi)+f^3(\mu)}{2f^3(\mu)}\right\} d\mu \quad . \tag{A.4.20}$$

while

$$Q^{II} = \frac{1}{6} \int_\xi^\eta \frac{f^2(\eta)f'(\eta)}{f(\mu)[f^3(\eta)+f^3(\mu)]} \ln\left\{\frac{f^3(\xi)+f^3(\eta)}{f^3(\eta)+f^3(\mu)}\right\} d\mu$$

$$- \frac{1}{6} \int_\xi^\eta \frac{f^2(\eta)f'(\eta)}{f(\mu)[f^3(\mu)-f^3(\eta)]} \ln\left\{\frac{f^3(\mu)+f^3(\eta)}{2f^3(\mu)} \cdot \frac{f^3(\mu)+f^3(\xi)}{f^3(\xi)+f^3(\eta)}\right\} d\mu \quad . \tag{A.4.21}$$

The only difficulty is making our estimate arise from the second term in each case. We write, using (A.4.20) as an illustration,

$$\frac{1}{f^3(\mu)-f^3(\xi)} \quad \ell n \left\{ \frac{f^3(\mu)+f^3(\eta)}{f^3(\xi)+f^3(\eta)} \cdot \frac{f^3(\xi)+f^3(\mu)}{2f^3(\mu)} \right\}$$

$$= \frac{1}{f^3(\mu)-f^3(\xi)} \left\{ \left[\frac{f^3(\mu)-f^3(\xi)}{f^3(\xi)+f^3(\eta)} + \frac{f^3(\mu)-f^3(\xi)}{f^3(\xi)+f^3(\eta)} \right]^2 \cdot \left(\frac{-1}{(1+\lambda)^2} \right) \right.$$

$$\left. - \left[\frac{f^3(\mu)-f^3(\xi)}{2f^3(\mu)} \right] + \left[\frac{f^3(\mu)-f^3(\xi)}{2f^3(\mu)} \right]^2 \left(\frac{1}{(1+\lambda_2)^2} \right) \right\} , \tag{A.4.22}$$

where $0 < \lambda_1, \lambda_2 < 1$. Then we have this equal to

$$\frac{1}{f^3(\xi)+f^3(\eta)} - \frac{1}{2f^3(\mu)} + 0(\varepsilon^2)$$

$$= \frac{\{f^3(\mu)-f^3(\xi)\} + \{f^3(\mu)-f^3(\eta)\}}{2f^3(\mu) \quad \{f^3(\xi)+f^3(\eta)\}} + 0(\varepsilon^2) \quad . \tag{A.4.23}$$

It is now a simple matter to see that

$$\begin{aligned} |P^{II}| &\leq \frac{\varepsilon^2}{2} \; K^* \; |\xi-\eta| \quad , \\ |Q^{II}| &\leq \frac{\varepsilon^2}{2} \; K^* \; |\xi-\eta| \quad , \end{aligned} \tag{A.4.24}$$

for some $K^* > 0$. Taking $\overline{K} = \max\,[K, K^*]$ it follows that

$$\begin{aligned} |p_2-p_1| &< \overline{K} \; \varepsilon^2 \; |\xi-\eta| \quad , \\ |q_2-q_1| &< \overline{K} \; \varepsilon^2 \; |\xi-\eta| \quad , \end{aligned} \tag{A.4.25}$$

The remainder of our argument will now follow in a standard

way. Let

$$\mu = \max_{x,y} \left| \frac{a\Pi^2 \sin \Pi x \ f(x)}{f^3(x)+f^3(y)} \right| \quad . \tag{A.4.26}$$

Then

$$\begin{aligned} |p_{n+1}-p_n| &\leq \varepsilon\mu \left| \int_\xi^\eta (|p_n-p_{n-1}| + |q_n-q_{n-1}|)d\eta \right| , \quad \xi \ \text{fixed} , \\ |q_{n+1}-q_n| &\leq \varepsilon\mu \left| \int_\xi^\eta (|p_n-p_{n-1}| + |q_n-q_{n-1}|)d\xi \right| , \quad \eta \ \text{fixed} . \end{aligned} \tag{A.4.27}$$

We then obtain for $n > 1$,

$$|p_n-p_{n-1}| \leq (2\mu)^{n-2} K \varepsilon^n \frac{|\xi-\eta|^{n-1}}{(n-1)!} \quad , \tag{A.4.28}$$

and so

$$\begin{aligned} |p_n-p_1| &\leq \sum_{k=2}^{\infty} (p_k-p_{k-1}) = K \sum_{k=2}^{\infty} (2\mu)^{k-2} \varepsilon^k \frac{|\xi-\eta|^{k-1}}{(k-1)!} \\ &\leq \frac{K\varepsilon}{2\mu} \sum_{k=2}^{\infty} (2\mu\varepsilon)^{k-1} \frac{|\xi-\eta|^{k-1}}{(k-1)!} \\ &= \frac{K\varepsilon}{2} \sum_{n=1}^{\infty} \frac{[(2\mu\varepsilon)|\xi-\eta|]^n}{m!} \\ &= \frac{K\varepsilon}{2} \{e^{2\mu\varepsilon|\xi-\eta|} - 1\} \quad . \end{aligned} \tag{A.4.29}$$

As we bound $\varepsilon|\xi-\eta|$, it follows that $|p_n-p_1| < \varepsilon D$ for a finite constant $D > 0$, and our conjecture that to order ε , the zeros of t_ξ are given by the zeros of $T_\xi + p_1$ is correct.

A simple calculation shows that

$$T_\xi + p_1 = -\frac{1}{2} - \varepsilon \frac{a\Pi^2}{16} (\eta-\xi) \sin \Pi\eta + O(\varepsilon) \quad ,$$

which is in agreement with the results of Chapter IV.

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